MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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COOPERATIVE SCATTERING BY DIELECTRIC SPHERES

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Group 61

TECHNICAL NOTE 1967-31

6 JULY 1967

LEXINGTON MASSACHUSETTS

ABSTRACT

The problem of scattering of electromagnetic waves by a small number of closely spaced dielectric spheres is considered as a boundary value problem. The solution to this problem is obtained in a series form using partial spherical vector waves. An approximate solution is also obtained for spheres separated sufficiently far for waves scattered by one sphere and incident on another to be considered plane waves with an amplitude given by the solution to the single scattering problem. The use of both solutions is discussed.

Accepted for the Air Force
Franklin C. Hudson,
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PART I - THEORY

INTRODUCTION

The solution to the problem of multiple scattering by an ensemble of dielectric spheres is important to the understanding of the propagation of millimeter electromagnetic waves through a rain environment. Attempts at solving the problem have been made using the incoherent, geometrical optics formulation of radiative transfer theory (Chandrasekhar, 1950), a coherent physical optics formulation as applied to scalar fields by Fikioris (1966), or using a coherent, full-wave treatment as applied to scalar fields by Fikioris and Waterman (1964). The problems of using a full-wave solution are formidable. This study is an investigation of the conditions under which a physical optics solution is valid. The method of investigation is that of comparing the results of both a full-wave and a physical-optics computation of the bistatic scattering cross section for a fixed configuration of a small number of spheres. The examples chosen for analysis were picked to be compared with the results of the experimental bistatic scattering cross-section measurements published by Moe and Angelakos (1961).

The problem of computing the backscatter cross section of a pair of spheres using a full-wave formalism has been considered by Trinks (1935) for small spheres and by Liang and Lo (1966) for spheres of the order of a wavelength in diameter. The results of Liang and Lo compare favorably with the experimental data.

The physical optics technique for solving the multiple scattering problem consists of using the bistatic scattering cross section (far-field value) to relate the incident and scattered waves and of assuming that the incident fields can be represented by plane waves. The full-wave technique refers to methods that use the full solution to the boundary value problems of multiple scattering. This technique as referred to above and as used in this study entails the expansion of the incident and scattered waves in partial spherical vector waves (PSVW). The expansion coefficients are determined so that the boundary conditions on the multiple spheres are satisfied. The basic difficulty in the use of this technique is in handling the translation addition formulas required for expressing the wave scattered by one sphere in the coordinate system of another.

This report is devoted to the derivation of the full wave and physical optics solutions to the multiple scattering problem for a fixed configuration of scatterers. Either solution is obtained in terms of a set of simultaneous equations for the determination of the coefficients of an infinite PSVW series. The conditions for obtaining a solution of the infinite set of equations by truncating the set of equations are investigated. The results show that for the physical optics case a solution is always possible using a truncated set of equations. The solution by truncation is also always possible for the full-wave case if only two spheres are used. For the full-wave case and more than two spheres the solution may not be possible. This case must be investigated further using the computer.

Review of the Mie solution for a single sphere

Time harmonic electromagnetic fields in source free space may be represented by a summation of partial spherical vector waves (Stratton, 1941) with the time dependence taken as $\,_j\underset{\sim}{E}(\underset{\sim}{r}_j, t) = \,_j\underset{\sim}{E}(\underset{\sim}{r}_j)\, e^{i\omega t}$.

The partial vector wave expansion is given by

$$\,_j\underset{\sim}{E} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{e=1}^{2} \,_ja^e_{nm} \,_j\underset{\sim}{v}^{e(\nu)}_{mn} (r_j, \vartheta_j, \varphi_j) \tag{1}$$

$$\,_j\underset{\sim}{H} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{e=1}^{2} \,_jb^e_{mn} \,_j\underset{\sim}{v}^{e(\nu)}_{mn} (r_j, \vartheta_j, \varphi_j) \tag{2}$$

in the j coordinate system using the notation given in Appendix A. The partial vector waves are related to those used by Stratton by

$$\,_j\underset{\sim}{v}^{1(\nu)}_{mn} = \,_j\underset{\sim}{m}^{(\nu)}_{mn} = \nabla \times [\psi^{(\nu)}_{nm} (r_j, \vartheta_j, \varphi_j)\, \underset{\sim}{r}_j]$$

$$\,_j\underset{\sim}{v}^{2(\nu)}_{mn} = \,_j\underset{\sim}{n}^{(\nu)}_{mn} = \frac{1}{k} \nabla \times \,_j\underset{\sim}{m}^{(\nu)}_{mn} = \frac{1}{k} \nabla \times \nabla \times [\psi^{(\nu)}_{nm} (r_j, \vartheta_j, \varphi_j]\, \underset{\sim}{r}_j)$$

where

$\psi^{(\nu)}_{nm} = z^{(\nu)}_n (kr_j) P^m_n (\cos\vartheta_j) e^{im\varphi_j}$, $z^{(\nu)}_n$ are spherical Bessel functions and P^m_n are associated Legendre functions of the first kind.

For the source free problem the ${}_j\underset{\sim}{v}^{e(\nu)}_{mn}$ PSVW form a complete orthogonal set. The fields, ${}_j\underset{\sim}{H}$ and ${}_j\underset{\sim}{E}$ can be formally represented by a column matrix of the expansion coefficients

$$ {}_j\underline{a} = \left\{ {}_j a^e_{mn} \right\} \; ; \; {}_j\underline{b} = \left\{ {}_j b^e_{mn} \right\} \quad . $$

Using this representation, the Mie solution to scattering by a dielectric sphere is found as a diagonal matrix relating the matricies for the incident and scattered fields.

The problem of scattering by sphere "j" is solved by using the boundary conditions to determine the Mie scattering matrix. The field exterior to the sphere is separated into an incident and scattered wave as

$$ {}_j\underset{\sim}{E} = {}^i_j\underset{\sim}{E} + {}^s_j\underset{\sim}{E} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{e=1}^{2} \left({}^i_j a^e_{mn} \, {}_j\underset{\sim}{v}^{e(1)}_{mn1} + {}^s_j a^e_{mn} \, {}_j\underset{\sim}{v}^{e(4)}_{mn1} \right) \qquad r_j \geq \rho_j \quad (3) $$

where ρ_j = radius of sphere j, the super prefix i represents the incident field with a radial function denoted by (1) that is finite at the origin, the super prefix s represents the scattered field with a radial function denoted by (4) that represents outgoing waves as required by the Sommerfeld radiation condition.

The internal field is given in a similar way by

$$ {}_j\underset{\sim}{E} = {}_j^t\underset{\sim}{E} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{e=1}^{2} {}_j^t a_{mn}^e \; {}_n\underset{\sim}{v}_{mn2}^{e(1)} \qquad r_j \le \rho_j \tag{4} $$

with a radial dependence that is finite at the origin. The boundary conditions for the fields at the surface of the sphere are given by

$$ \hat{n}_j \times ({}_j^i\underset{\sim}{E} + {}_j^s\underset{\sim}{E} - {}_j^t\underset{\sim}{E}) = 0 \tag{5} $$

$$ \hat{n}_j \times ({}_j^i H + {}_j^s H - {}_j^t H) = 0 $$

where $\hat{n}_j$ is the unit outward normal to the surface of the sphere at coördinate $(\rho_j, \vartheta_j, \varphi_j)$. The boundary conditions reduce to a set of equations for the PSVW amplitudes.

$$ \hat{n}_j \times \left(\sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{e=1}^{2} {}_j^i a_{mn}^e \; \underset{\sim}{v}_{mn1}^{e(1)} + {}_j^s a_{mn}^e \; \underset{\sim}{v}_{mn1}^{e(4)} - {}_j^t a_{mn}^e \; \underset{\sim}{v}_{mn2}^{e(1)} \right) = 0 $$

where the 1, 2 subscripts on $\underset{\sim}{v}$ denote the wave number required for use in the radial function, k_1 exterior to the sphere and k_2 interior to the sphere. Using the orthogonality properties of $\underset{\sim}{v}$ as given in Appendix B, the boundary condition equations can be reduced to a set of algebraic equations

$$\left({}_{j}\underset{\sim}{v}^{\ell(1)}_{qp1},\ {}_{j}\hat{n} \times \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{e=1}^{2} \left[{}^{i}_{j}a^{e}_{mn}\ {}_{j}\underset{\sim}{v}^{e(1)}_{mn1} + {}^{s}_{j}a^{e}_{mn}\ {}_{j}\underset{\sim}{v}^{e(4)}_{mn1} - {}^{t}_{j}a^{e}_{mn}\ {}_{j}\underset{\sim}{v}^{e(1)}_{mn2} \right] \right) = 0$$

$$= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{e=1}^{2} \left[{}^{i}_{j}a^{e}_{mn} \left({}_{j}\underset{\sim}{v}^{\ell(1)}_{qp1},\ \hat{r}_{j} \times {}_{j}v^{e(1)}_{mn} \right) + \right.$$

$$+ {}^{s}_{j}a^{e}_{mn} \left({}_{j}\underset{\sim}{v}^{\ell(1)}_{qp1},\ \hat{r}_{j} \times {}_{j}\underset{\sim}{v}^{e(4)}_{mn1} \right) -$$

$$\left. - {}^{t}_{j}a^{e}_{mn} \left({}_{j}\underset{\sim}{v}^{\ell(1)}_{qp1},\ \hat{r}_{j} \times {}_{j}\underset{\sim}{v}^{e(1)}_{mn2} \right) \right] .$$

now $$\left({}_{j}\underset{\sim}{v}^{1(\kappa)}_{qpu},\ \hat{r}_{j} \times {}_{j}\underset{\sim}{v}^{1(\nu)}_{mnw} \right) = \left({}_{j}\underset{\sim}{v}^{2(\kappa)}_{qpu},\ \hat{r}_{j} \times {}_{j}\underset{\sim}{v}^{2(\nu)}_{mnw} \right) = 0$$

$$\left({}_{j}\underset{\sim}{v}^{1(\kappa)}_{qpu},\ \hat{r}_{j} \times {}_{j}\underset{\sim}{v}^{2(\nu)}_{mnw} \right) = \frac{z^{(\kappa)*}_{p}(k_{u} r_{j})}{k_{w} r_{j}} \frac{\partial}{\partial r_{j}} \left[r_{j} z^{(\nu)}_{p}(k_{w} r_{j}) \right] \frac{4\pi p(p+1)}{(2p+1)} \frac{(p+q)!}{(p-q)!} \delta_{mq} \delta_{np}$$

$$\left({}_{j}\underset{\sim}{v}^{2(\kappa)}_{qpu},\ \hat{r}_{j} \times {}_{j}\underset{\sim}{v}^{1(\nu)}_{mnw} \right) = \frac{z^{(\nu)}_{p}(k_{w} r_{j})}{k_{u} r_{j}} \frac{\partial}{\partial r_{j}} \left[r_{j} z^{(\kappa)*}_{p}(k_{u} r_{j}) \right] \frac{4\pi p(p+1)}{(2p+1)} \frac{(p+q)!}{(p-q)!} \delta_{mq} \delta_{np}$$

Therefore

$${}^{i}_{j}a^{1}_{qp}\, j_{p}(k_{1}\rho_{j}) + {}^{s}_{j}a^{1}_{qp}\, h^{(2)}_{p}(k_{1}\rho_{j}) - {}^{t}_{j}a^{1}_{qp}\, j_{p}(k_{2}\rho_{j}) = 0$$

$$\left[{}_j^{i}a_{qp}^{2} \frac{\frac{\partial}{\partial r_j}\left[r_j j_p(k_1 r_j)\right]}{k_1 r_j} + {}_j^{s}a_{qp}^{2} \frac{\frac{\partial}{\partial r_j}\left[r_j h_p^{(2)}(k_1 r_j)\right]}{k_1 r_j} - {}_j^{t}a_{qp}^{2} \frac{\frac{\partial}{\partial r_j}\left[r_j j_p(k_2 r_j)\right]}{k_2 r_j} \right]_{r_j=\rho_j} = 0 \; .$$

The boundary conditions for ${}_j\underset{\sim}{H}$ yield identical equations with a replaced by b. The equations can be further reduced by the relationship between ${}_j\underset{\sim}{E}$ and ${}_j\underset{\sim}{H}$. From Maxwell's equations for a source free space,

$$i\omega\mu \, {}_j\underset{\sim}{H} = -\nabla \times {}_j\underset{\sim}{E} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{e=1}^{2} {}_j a_{mn}^{e} \nabla \times {}_j\underset{\sim}{v}_{mn}^{e(\nu)} \; .$$

By the basic properties of ${}_j\underset{\sim}{m}_{mn}$ and ${}_j\underset{\sim}{n}_{mn}$ as given in Stratton,

$$\underset{\sim}{v}_{mn}^{3-e(\nu)} = \frac{1}{k} \nabla \times \underset{\sim}{v}_{mn}^{e(\nu)}$$

therefore, $$i\omega\mu \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{e=1}^{2} {}_j b_{mn}^{e} \, {}_j\underset{\sim}{v}_{mn}^{e(\nu)} = -k \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{e=1}^{2} {}_j a_{mn}^{e} \, {}_j\underset{\sim}{v}_{mn}^{3-e(\nu)} \; .$$

Using the orthogonality properties of $\underset{\sim}{v}$,

$$i\omega\mu \, {}_j b_{mn}^{e} = -k \, {}_j a_{mn}^{3-e} \tag{6}$$

or

$$ {}_j b^{e}_{mn} = \frac{ik}{\omega\mu}\, {}_j a^{3-e}_{mn} \quad . $$

From this, four sets of simultaneous equations are available for determining the Mie scattering matrix elements. Using the equations for ${}_j\underline{b}$, find

$$ \frac{ik_1}{\omega\mu_1}\, {}^s_j a^2_{qp}\, h^{(2)}_p(k_1\rho_j) - \frac{ik_2}{\omega\mu_2}\, {}^t_j a^2_{qp}\, j_p(k_2\rho_j) = -\frac{ik_1}{\omega\mu_1}\, {}^i_j a^2_{qp}\, j_p(k_1\rho_j) $$

$$ \frac{ik_1}{\omega\mu_1}\, {}^s_j a^1_{qp}\, \frac{1}{k_1\rho_j}\left[\frac{\partial}{\partial r}\left[r h^{(2)}_p(k_1 r)\right]\right]_{r=\rho_j} - \frac{ik_2}{\omega\mu_2}\, {}^t_j a^1_{qp}\, \frac{1}{k_2\rho_j}\left[\frac{\partial}{\partial r}\left[r j_p(k_2 r)\right]\right]_{r=\rho_j} = $$

$$ = -\frac{ik_1}{\omega\mu_2}\, {}^i_j a^1_{qp}\, \frac{1}{k_1\rho_j}\left[\frac{\partial}{\partial r}\left[r j_p(k_1 r)\right]\right]_{r=\rho_j} \quad . $$

The equations for ${}_j\underline{a}$ can be restated as

$$ {}^s_j a^1_{qp}\, h^{(2)}_p(k_1\rho_j) - {}^t_j a^1_{qp}\, j_p(k_2\rho_j) = -\, {}^i_j a^1_{qp}\, j_p(k_1\rho_j) $$

$$ {}^s_j a^2_{qp}\, \frac{1}{k_1\rho_j}\left[\frac{\partial}{\partial r}\left[r h^{(2)}_p(k_1 r)\right]\right]_{r=\rho_j} - {}^t_j a^2_{qp}\, \frac{1}{k_2\rho_j}\left[\frac{\partial}{\partial r}\left[r j_p(k_2 r)\right]\right]_{r=\rho_j} = $$

$$ = -\, {}^i_j a^2_{qp}\, \frac{\rho}{k_1\rho_j}\left[\frac{\partial}{\partial r}\left[r j_p(k_1 r)\right]\right]_{r=\rho_j} \quad . $$

These equations can be solved for ${}^{s}_{j}a^{t}_{qp}$ by Cramer's rule with $\mu_1 = \mu_2$

$$
{}^{s}_{j}a^{1}_{qp} = \frac{\begin{vmatrix} -j_p(k_1\rho_j) & -j_p(k_2\rho_j) \\ -\left[\frac{\partial}{\partial r}[rj_p(k_1 r)]\right]_{r=\rho_j} & -\left[\frac{\partial}{\partial r}[rj_p(k_2 r)]\right]_{r=\rho_j} \end{vmatrix}}{\begin{vmatrix} h_p^{(2)}(k_1\rho_j) & -j_p(k_2\rho_j) \\ \left[\frac{\partial}{\partial r}[rh_p^{(2)}(k_1 r)]\right]_{r=\rho_j} & -\left[\frac{\partial}{\partial r}[rj_p(k_2 r)]\right]_{r=\rho_j} \end{vmatrix}} \; {}^{i}_{j}a^{1}_{qp} = M^{1}_{p}(k_1, k_2, \rho_j)\, {}^{i}_{j}a^{1}_{qp} \tag{7}
$$

$$
{}^{s}_{j}a^{2}_{qp} = \frac{\begin{vmatrix} -k_1 j_p(k_1\rho_j) & -k_2 j_p(k_2\rho_j) \\ -\frac{1}{k_1}\left[\frac{\partial}{\partial r}[j_p(k_1 r)]\right]_{r=\rho_j} & -\frac{1}{k_2}\left[\frac{\partial}{\partial r}[rj_p(k_2 r)]\right]_{r=\rho_j} \end{vmatrix}}{\begin{vmatrix} k_1 h_p^{(2)}(k_1\rho_j) & -k_2 j_p(k_2\rho_j) \\ \frac{1}{k_1}\left[\frac{\partial}{\partial r}[rh_p^{(2)}(k_1 r)]\right]_{r=\rho_j} & -\frac{1}{k_2}\left[\frac{\partial}{\partial r}[rj_p(k_2 r)]\right]_{r=\rho_j} \end{vmatrix}} \; {}^{i}_{j}a^{2}_{qp} = M^{2}_{p}(k_1, k_2, \rho_j)\, {}^{i}_{j}a^{2}_{qp} \; . \tag{8}
$$

The matrix equation for the single sphere is formed from the $M^{t}_{p}(k_1, k_2, \rho_j)$ elements.

$$
{}^{s}_{j}\underline{a} = {}_{j}\underline{\underline{M}}\; {}^{i}_{j}\underline{a} \tag{9}
$$

where ${}_{j}\underline{\underline{M}}$ is a diagonal matrix ${}_{j}\underline{\underline{M}} = \left(M^{t}_{n}(k_1, k_2, \rho_j)\, \delta_{et}\delta_{np}\delta_{mq}\right)$.

The Mie scattering matrix can be used for any incident field ${}^{i}_{j}\underline{a}$ to generate the scattered field ${}^{s}_{j}\underline{a}$. This relationship holds only in the coordinate system j for any azimuthal index. The values of M^{e}_{p} given above are related to those of Van de Hulst (1957, Chapter 9) by

$$M^{1}_{p} = -b_{p}$$

$$M^{2}_{p} = -a_{p} \quad .$$

<u>Multiple Sphere Boundary Value Problem</u>

The multiple sphere problem is solved by extending the solution for a single sphere to a configuration of spheres. Consider a fixed configuration of J spheres. Let the scattered wave from each sphere be expressed in partial spherical vector waves in a coordinate system centered on that sphere. The total electromagnetic field exterior to each sphere is given as a sum of the incident wave and the waves scattered by each sphere.

$${}_{j}\underset{\sim}{E} = {}^{i}_{j}\underset{\sim}{E} + \sum_{\ell=1}^{J} {}^{s}_{\ell}\underset{\sim}{E} \quad . \tag{10}$$

This total exterior field must obey the boundary conditions on each sphere simultaneously. The solution is obtained by identifying the waves scattered by all spheres other than j with the incident wave on sphere j. In the region exterior to the scattering sphere, the scattered wave can be represented using partial vector waves in another coordinate system since the scattered wave satisfies the source free vector wave equation.

$$
{}_{\ell}\underset{\sim}{v}^{e(\nu)}_{qp} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{t=1}^{2} {}^{j}_{\ell}C^{eqp(\nu)}_{tmn} \; {}_{j}\underset{\sim}{v}^{t(1)}_{mn} \quad . \tag{11}
$$

This relationship called a partial spherical vector wave addition theorem is given by Cruzan (1962) and by Stein (1961) and is valid for radial distances in the new coordinate system less than the translation distance. The addition theorem may be decomposed into two parts, a rotation as given by Edmonds (1957) and a transformation of the coordinate system along the $\vartheta = 0$ azis. These two specialized operations can be combined to give the general one above

$$
{}^{j}_{\ell}C^{eqp(\nu)}_{tmn} = \sum_{s} {}_{\ell}R^{eqp}_{esp}\,(\alpha,\beta,0)\,{}^{j}_{\ell}\Omega^{esp(\nu)}_{tsn}\,(kd)\;{}_{j}R^{tsn}_{tmn}\,(0,\beta,-\alpha)
$$

where $-p \leq s \leq p$ and $-n \leq s \leq n$ and the summation is over all allowed values of s.

${}_{j}R^{eqp}_{esp}\,(\alpha,\beta,\gamma)$ describes a rotation of the coordinate system with origin at j through the Euler angles α, β, γ and ${}^{j}_{\ell}\Omega^{esp(\nu)}_{tsn}$ (kd) describes a translation of the coordinate system through a distance d along the direction $\vartheta_j = 0$. The two addition theorems are given in Appendix C. The transformation can be considered as a matrix operation as

$$
{}_{\ell}^{s}\underset{\sim}{E} = \sum_{p=0}^{\infty} \sum_{q=-p}^{p} \sum_{e=1}^{2} {}_{\ell}^{s}a_{qp}^{e} \; {}_{\ell}\underset{\sim}{v}_{qp}^{e(4)}
$$

$$
= \sum_{p=0}^{\infty} \sum_{q=-p}^{p} \sum_{e=1}^{2} {}_{\ell}^{s}a_{qp}^{e} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{t=1}^{2} {}_{\ell}^{j}C_{tmn}^{eqp(4)} \; {}_{j}\underset{\sim}{v}_{mn}^{t(1)}
$$

$$
= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{t=1}^{2} {}_{\ell}^{j}d_{mn}^{t} \; {}_{j}\underset{\sim}{v}_{mn}^{t(1)}
$$

where the interchange of summation is justified by Friedman and Russek (1954). The boundary value problem then reduces to

$$
{}_{j}^{s}\underline{a} = {}_{j}\underline{\underline{M}} \left({}_{j}^{i}\underline{a} + \sum_{\substack{\ell=1 \\ \ell \neq j}}^{J} {}_{j}^{\ell}\underline{d} \right)
$$

$$
= {}_{j}\underline{\underline{M}} \left({}_{j}^{i}\underline{a} + \sum_{\substack{\ell=1 \\ \ell \neq j}}^{J} {}_{j}^{\ell}\underline{\underline{C}} \; {}_{\ell}^{s}\underline{a} \right) \qquad j = 1, 2, \ldots J. \tag{12}
$$

The solution is obtained by a simultaneous solution of the J matrix equations to determine the J scattered waves given by ${}_{j}^{s}\underline{a}$.

The final solution to the multiple sphere, fixed configuration scattering problem is best given by the ${}_{j}^{s}\underline{a}$ all expressed in a single coordinate system so that the bistatic scattering cross section can be readily identified. The ${}^{s}_{j}\underline{a}$ can be converted back to a single coordinate system using the PSVW addition theorem for radial distances greater than the translation distance

$$ {}_{\ell}\underset{\sim}{v}{}^{e(\nu)}_{qp} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{t=1}^{2} {}^{j}_{\ell}f^{eqp(1)}_{tmn} \; {}_{j}\underset{\sim}{v}{}^{t(\nu)}_{mn} $$

which as for $\underline{\underline{C}}$ above can be reduced to a rotation and a translation. Using addition theorems

$$ {}^{s}\underline{a} = {}^{j}\underline{\underline{F}} \; {}^{s}_{j}\underline{a} \quad . $$

The solution to the scattering problem then is given by

$$ {}^{s}\underline{a} = \sum_{j=1}^{J} {}^{j}\underline{\underline{F}} \; {}^{s}_{j}\underline{a} \tag{13} $$

a single column matrix representing the total scattered field of the configuration of spheres.

Solution of the Boundary Value Equations — Two-Sphere Case

The solution of the multiple scattering problem depends upon the simultaneous solution of the J matrix equations. The elements of these equations form semi-infinite matrices since they represent the coefficients of infinite sets of partial spherical vector wave solutions to the vector wave equations. These equations must be solved by approximate techniques. The criterion for obtaining a solution by truncating the matrices and solving the truncated linear equations using algebra is given in Appendix D. To apply this technique of reducing the problem to an algebraic one which can be solved on a large scale computer, the behavior of the elements of the matrix equations must be ascertained.

For the two sphere scattering problem, the simultaneous matrix equations (12) are given by

$$ {}^{s}_{1}\underline{a} = {}_{1}\underline{\underline{M}}\left({}^{i}_{1}\underline{a} + {}^{2}_{1}\underline{\underline{C}}\,{}^{s}_{2}\underline{a}\right) $$

$$ {}^{s}_{2}\underline{a} = {}_{2}\underline{\underline{M}}\left({}^{i}_{2}\underline{a} + {}^{1}_{2}\underline{\underline{C}}\,{}^{s}_{1}\underline{a}\right) \quad . $$

These equations can be combined to give

$$ {}^{s}_{1}\underline{a} = {}_{1}\underline{\underline{M}}\left({}^{i}_{1}\underline{a} + {}^{2}_{1}\underline{\underline{C}}\,{}_{2}\underline{\underline{M}}\left[{}^{i}_{2}\underline{a} + {}^{1}_{2}\underline{\underline{C}}\,{}^{s}_{1}\underline{a}\right]\right) $$

$$ = {}_{1}\underline{\underline{M}}\,{}^{2}_{1}\underline{\underline{C}}\,{}_{2}\underline{\underline{M}}\,{}^{1}_{2}\underline{\underline{C}}\,{}^{s}_{1}\underline{a} + {}_{1}\underline{\underline{M}}\,{}^{i}_{1}\underline{a} + {}_{1}\underline{\underline{M}}\,{}^{2}_{1}\underline{\underline{C}}\,{}_{2}\underline{\underline{M}}\,{}^{i}_{2}\underline{a} $$

$$ \left(\underline{1} - {}_{1}\underline{\underline{M}}\,{}^{2}_{1}\underline{\underline{C}}\,{}_{2}\underline{\underline{M}}\,{}^{1}_{2}\underline{\underline{C}}\right){}^{s}_{1}\underline{a} = {}_{1}\underline{\underline{M}}\left({}^{i}_{1}\underline{a} + {}^{2}_{1}\underline{\underline{C}}\,{}_{2}\underline{\underline{M}}\,{}^{i}_{2}\underline{a}\right) $$

or

$$ \left(\underline{\underline{1}} - {}_{1}\underline{\underline{S}}\right){}^{s}_{1}\underline{a} = {}_{1}\underline{B} \quad . \tag{14} $$

This set of equations may be solved by using a truncated set of equations if the conditions on S_{ij} and B_{i}, as specified in Appendix D, are met. To examine the applicability, the behavior of the ${}_{i}\underline{\underline{M}}\,{}^{j}_{i}\underline{\underline{C}}$ matrix elements must be determined.

The Mie coefficients can be investigated using the forms given above

$$M_n^1 = -\frac{j_n(k_1\rho)\left[\frac{\partial}{\partial r}[rj_n(k_2r)]\right]_{r=\rho} - j_n(k_2\rho)\left[\frac{\partial}{\partial r}[rj_n(k_1r)]\right]_{r=\rho}}{h_n^{(2)}(k_1\rho)\left[\frac{\partial}{\partial r}[rj_n(k_2r)]\right]_{r=\rho} - j_n(k_2\rho)\left[\frac{\partial}{\partial r}[rh_n^{(2)}(k_1r)]\right]_{r=\rho}}$$

$$= -\frac{k_1 j_{n+1}(k_1\rho)\, j_n(k_2\rho) - k_2 j_{n+1}(k_2\rho)\, j_n(k_1\rho)}{k_1 h_{n+1}^{(2)}(k_1\rho)\, j_n(k_2\rho) - k_2 j_{n+1}(k_2\rho)\, h_n^{(2)}(k_1\rho)} \quad .$$

The behavior of M_n^1 for large n and fixed $k_1\rho$, $k_2\rho$ can be estimated from the asymptotic forms of the spherical Bessel function.

$$j_n(k_1\rho) \sim \frac{1}{(2n+1)} \sqrt{\frac{e}{2}} \left(\frac{ek_1\rho}{2n+1}\right)^n \qquad n \to \infty$$

$$h_n^{(2)}(k_1\rho) \sim -in_n(k_1\rho) \sim \frac{-i}{k_1\rho} \sqrt{\frac{2}{e}} \left(\frac{ek_1\rho}{2n+1}\right)^{-n} \qquad n \to \infty$$

Using these forms, $j_n(k_1\rho) > j_{n+1}(k_1\rho)$ and $h_{n+1}^{(2)}(k_1\rho) > h_n^{(2)}(k_1\rho)$. For $|k_2| > k_1$, the terms in $j_n(k_1\rho)$ and $h_{n+1}^{(2)}(k_1\rho)$ contribute most to the asymptotic form of M_n^1

$$M_n^1 \sim \frac{k_2 j_{n+1}(k_2\rho) j_n(k_1\rho)}{k_1 j_n(k_2\rho) h_{n+1}^{(2)}(k_1\rho)} \sim \frac{ek_2^2\rho\,(2n+1)^{n+1} j_n(k_1\rho)}{k_1 (2n+3)^{n+2} h_{n+1}^{(2)}(k_1\rho)}$$

$$\left|M_n^1\right| \sim \frac{(ek_2\rho)^2 (ek_1\rho)^{2n+1}}{2(2n+3)^{2n+2}} = \left(\frac{ek_2\rho}{2n+3}\right)^2 \left(\frac{ek_1\rho}{2n+3}\right)^{2n} \left(\frac{ek_1\rho}{2}\right) \tag{15}$$

In a similar fashion, the asymptotic form of M_n^2 is given as

$$\left|M_n^2\right| \sim \frac{(n+1)}{2(2n+1}\left(\frac{k_1}{k_2}\right)(k_1 - k_2)\left(\frac{ek_1\rho}{2n+3}\right)^{n+1}\left(\frac{ek_1\rho}{2n+1}\right)^n \sim \frac{(n+1)(ek_1\rho)^{2n+1}}{2(2n+1)^{n+1}(2n+3)^{n+1}} \tag{16}$$

In both cases, for large n the Mie coefficient terms decrease as

$$M_n \propto \left(\frac{ek_1\rho}{2n}\right)^{2n+1} \qquad n \gg k_1\rho \quad .$$

The elements of the C matrix can be estimated using the formulas given in Appendix C. For two spheres, the only addition theorems to consider are

$${}^1_2\underline{\underline{C}} = {}^1_2\underline{\underline{\Omega}} \; ; \; {}^2_1\underline{\underline{C}} = {}_1\underline{\underline{R}}(0,\pi,0)\, {}^2_1\underline{\underline{\Omega}}\, {}_2\underline{\underline{R}}(0,\pi,0) \quad .$$

The elements of the ${}^2_1\underline{\underline{\Omega}}$ matrix are given by

$$\Omega_{emu}^{tmn(\kappa)} = T_{un}^{(m,\kappa)} + \frac{k_1 d\,(u+m+1)}{(2u+3)(u+1)} T_{u+1n}^{(m,\kappa)} + \frac{k_1 d\,(u-m)}{u(2u-1)} T_{u-1n}^{(m,\kappa)} \qquad e=t$$

$$= \frac{ik_1 d\; T_{un}^{(m,\kappa)}}{u(u+1)}, \qquad e \neq t \quad .$$

The dependence of Ω can best be studied using T as

$$T_{un}^{(m,\kappa)} = \sum_{p=|n-u|}^{n+u} i^{u+p-n}\, G_{pu}^{n}\, H_{pu}^{n0}\, H_{pu}^{nm}\, z_p^{(\kappa)}(k_1 d)$$

$$G_{pu}^{n} = \frac{(2p+1)(2u+1)}{(2n+1}\left[\frac{(p+n-u)!\,(p-n+u)!\,(-p+n+u)!}{(p+n+u+1)!}\right]$$

$$H_{pu}^{nm} = \sum_{s} (-1)^s \binom{n+m}{s}\binom{u-m}{p-s}\binom{p}{u-n+s} \quad .$$

The maximum of the $H_{pu}^{nm}\, H_{pu}^{n0}$ product is given as

$$H_{pu}^{nm}\, H_{pu}^{n0} \sim \binom{2n}{n}$$

for $\quad p=n+u,\ n=u,$ both $n+u$ large, and $m=0$

$\quad p=n,\ n\gg u$ and $m=n$

or $\quad \rho=u,\ u\gg n$ and $m=n$

For all other cases, this product is less. The resultant T then is given by

$$T_{un}^{(0,4)} \sim \frac{(n+u)!^2}{(n!)^2(u!)^2}\,\frac{(2n)!(2u)!}{(2n+2u)!}\binom{2n}{2u}\, h_{n+u}^{(2)}(k_1 d); \quad n=u\gg 1$$

$$T_{un}^{(n,4)} \sim \frac{(2n)!\,(u!)^2 2u}{(n!)^2} h_n^{(2)}(k_1 d); \qquad n \gg u$$

$$\sim (2n)!\, 2u^2 h_u^{(2)}(k_1 d); \qquad u \gg n \quad .$$

By Sterling's formula

$$n! = \sqrt{2\pi n}\, n^n e^{-n}$$

Therefore, $$T_{un}^{(0,4)} \sim \frac{n\sqrt{n+u}}{u\sqrt{\pi}\sqrt{nu}} h_{n+u}^{(2)}(k_1 d) \qquad n=u\gg 1$$

$$T_{un}^{(n,4)} \sim \frac{u u!\, 2^{2n+1}}{\sqrt{\pi n}} h_n^{(2)}(k_1 d) \qquad n \gg u$$

$$\sim (2n)!\, 2u^2 h_u^{(2)}(k_1 d) \qquad u \gg n \quad .$$

By using the asymptotic expansion for $h_{n+u}^{(2)}(k_1 d)$ for $n+u \gg 1$ and taking the expression for m=0 since, for a fixed maximum n and u this gives the largest value, the estimate of magnitude of the elements of $T_{un}^{(m,4)}$ can be given by

$$\left| T_{un}^{(m,4)} \right| \sim \sqrt{\frac{(n+u)\,e}{2\pi n u}}\; \frac{1}{k_1 d}\, \frac{n}{u} \left(\frac{2n+2u}{e k_1 d} \right)^{n+u} ; \qquad |m| \ll n \tag{17}$$

The expression for Ω has the same order of magnitude as T, therefore the asymptotic expression for the $\underline{\underline{C}}$ elements for large order can be given by

$$\left|{}_2^1C_{tmu}^{emn}\right| \sim \left(\frac{n}{u}\right)\sqrt{\frac{(n+u)\,e}{2\pi nu}}\;\frac{2^{n+u}}{k_1 d}\left(\frac{n+u}{2k_1 d}\right)^{n+u}$$

and

$$\left|{}_1^2C_{t\mu n}^{emn}\right| = \sum_s {}_1R_{esn}^{emn}(0,\pi,0)\;{}_1^2\Omega_{tsu}^{esn}\;{}_2R_{t\mu u}^{tsu}(0,\pi,0)$$

$$= \frac{(n+m)!}{(n-m)!}\;{}_1^2\Omega_{t-mn}^{e-mn}\,\frac{(n-m)!}{(n+m)!}\;\delta_{m\mu} = {}_2^1C_{t-mu}^{e-mn}\;\delta_{m\mu} \quad .$$

The dependence of the magnitude of the $\underline{\underline{C}}$ elements on the index n is given by the expression for $\left|{}_2^1C_{tmu}^{emn}\right|$. The magnitude of the $\underline{\underline{S}}$ elements can now be estimated from the $\underline{\underline{M}}\,\underline{\underline{C}}$ product elements as

$$\left|M_n^1C_{1mn}^{1mn}\right| \sim \frac{(ek_2\rho)^2}{4uk_1 d}\left(\frac{ek_1\rho}{2n}\right)^{2n+1}\left(\frac{2n+2u}{ek_1 d}\right)^{n+u}\sqrt{\frac{(n+u)\,e}{2\pi nu}} \qquad |m| \ll n \text{ or } u \quad .$$

For the maximum value, n=u and the above product reduces to

$$\left|M_n^1C_{1mn}^{1mn}\right| \sim \frac{(ek_2\rho)^2}{n}\,\frac{e}{16n}\,\sqrt{\frac{e}{\pi n}}\left(\frac{2\rho}{d}\right)^{2n+1} \qquad |m| \ll n \qquad (19)$$

and

$$\left|M_n^2C_{2mn}^{2mn}\right| \sim \frac{e}{16n}\,\sqrt{\frac{e}{\pi n}}\left(\frac{2\rho}{d}\right)^{2n+1} \qquad |m| \ll n \quad . \qquad (20)$$

The elements of the $\underline{\underline{S}}$ matrix are less than or of the order of $S_{\rho mn}^{2mn}$ given by

$$S_{\rho mn}^{2mn} = \sum_{\nu} \sum_{t} M_n^1 C_{tm\nu}^{1mn} M_\nu^t C_{\rho mn}^{tm\nu}$$

$$\sim \frac{e^3}{16^2 n^3 \pi} \left(\frac{2\rho_1}{d}\right)^{2n+1} \left(\frac{2\rho_2}{d}\right)^{2n+1} \tag{21}$$

The elements of the $\underline{\underline{S}}$ matrix show that, for large separations between the two spheres, the elements decrease rapidly due to the factors in $2\rho_1/d$ and $2\rho_2/d$. If both spheres are of equal size and touch, these ratios are unity and the decrease in magnitude of the S elements with increasing n is much slower. This indicates that the multiple scattering contributions are much stronger for closely spaced spheres than for widely separated spheres. For the worst case of two equal touching spheres, the values of S elements will decrease with increasing n and, with a large scale computer the problem can be solved.

The analysis to be complete must also consider the B_n elements before a firm estimate of the behavior of the solution is obtained. The $\underline{B}$ matrix depends upon the $\underline{\underline{M}}\ \underline{\underline{C}}$ product and the standard Mie solution. The values for ${}_j\underline{a}^i$ are presumed to be known. For a plane wave they are given by an inverse n dependence (see Appendix B). The values of B_n for the plane waves therefore are less than the equivalent Mie matrix coefficient for the same n and the conclusion about the validity of the solution is unchanged.

The above considerations show that the requirements for solution by truncation of the multiple scattering matrix equations for two spheres are met for all possible sphere separations. Both the Mie coefficient and the translation matrix elements depend on the azimuthal index as a parameter only. This is due to the azimuthal symmetry of the problem when the translation is along the $\vartheta = 0$ direction. The matrix equations can be partitioned into sub-matrix equations one for each allowed value of m, the azimuthal index. The equations for each value of m can be solved independently of those for the other values of m. The reduced matrix equations then can be represented as

$$(\underline{\underline{1}} - \underline{\underline{S}}^{(m)})\ {}_1^s\underline{a}^{(m)} = {}_1\underline{B}^{(m)} \qquad -N \le m \le N \tag{22}$$

where the index m represents the parameter m and N is the largest n used in the truncated matrix and m, n are the indices of the ${}_1^s\underline{a}$ element ${}_1^s a^{e}_{mn}$.

The solution then is found by solving 2N + 1 sets of simultaneous equations, one for each value of m. For a plane wave incident along the axis of symmetry, only two sets with m = ± 1 are required.

<u>Solution of the Boundary Value Equations – Multiple Sphere Case</u>

The multiple sphere case is solved in the same manner as the two-sphere problem. The set of matrix equations must be reduced to a single equation with one unknown set of scattering coefficients. Unless the configuration of spheres has azimuthal symmetry, the general vector addition theorem for translation must be used.

$$ {}_{j}\underline{\underline{C}} = {}_{j}\underline{\underline{R}}\;{}^{i}_{j}\underline{\underline{\Omega}}\;{}_{i}\underline{\underline{R}} $$

The addition of more spheres doubly complicates the problem, one by requiring a simultaneous solution for all the n, m elements of the matrix instead of only n elements for a given value of m as in Eq. (22), and, two by increasing the number of matrix products required to specify ${}_{1}\underline{\underline{S}}$ and ${}_{1}\underline{B}$. For three spheres, these matrices are given by

$$ \begin{aligned} {}_{1}\underline{\underline{S}} = {} & {}_{3}\underline{\underline{M}}\,{}^{2}_{3}\underline{\underline{C}}\,{}_{2}\underline{\underline{M}}\,{}^{3}_{2}\underline{\underline{C}} + {}_{1}\underline{\underline{M}}\,{}^{2}_{1}\underline{\underline{C}}\,{}_{2}\underline{\underline{M}}\,{}^{1}_{2}\underline{\underline{C}} + {}_{1}\underline{\underline{M}}\,{}^{3}_{1}\underline{\underline{C}}\,{}_{3}\underline{\underline{M}}\,{}^{1}_{3}\underline{\underline{C}} \\ & + {}_{1}\underline{\underline{M}}\,{}^{3}_{1}\underline{\underline{C}}\,{}_{3}\underline{\underline{M}}\,{}^{2}_{3}\underline{\underline{C}}\,{}_{2}\underline{\underline{M}}\,{}^{1}_{2}\underline{\underline{C}} + {}_{1}\underline{\underline{M}}\,{}^{2}_{1}\underline{\underline{C}}\,{}_{2}\underline{\underline{M}}\,{}^{3}_{2}\underline{\underline{C}}\,{}_{3}\underline{\underline{M}}\,{}^{1}_{3}\underline{\underline{C}} \\ & + {}_{1}\underline{\underline{M}}\,{}^{2}_{1}\underline{\underline{C}}\,{}_{2}\underline{\underline{M}}\,{}^{3}_{2}\underline{\underline{C}}\,{}_{3}\underline{\underline{M}}\,{}^{2}_{3}\underline{\underline{C}}\,{}_{2}\underline{\underline{M}}\,{}^{1}_{2}\underline{\underline{C}} \\ & - {}_{3}\underline{\underline{M}}\,{}^{2}_{3}\underline{\underline{C}}\,{}_{2}\underline{\underline{M}}\,{}^{3}_{2}\underline{\underline{C}}\,{}_{1}\underline{\underline{M}}\,{}^{2}_{1}\underline{\underline{C}}\,{}_{2}\underline{\underline{M}}\,{}^{1}_{2}\underline{\underline{C}} \end{aligned} \qquad (23) $$

and

$$
\begin{aligned}
{}_1\underline{B} = {} & (\underline{\underline{1}} + {}_3\underline{\underline{M}}\;{}_3^2\underline{\underline{C}}\;{}_2\underline{\underline{M}}\;{}_2^3\underline{\underline{C}})\;{}_1\underline{\underline{M}}\;{}_1^i\underline{a} + ({}_1\underline{\underline{M}}\;{}_1^3\underline{\underline{C}} + {}_1\underline{\underline{M}}\;{}_1^2\underline{\underline{C}}\;{}_2\underline{\underline{M}}\;{}_2^3\underline{\underline{C}})\;{}_3\underline{\underline{M}}\;{}_3^i\underline{a} \\
& + ({}_1\underline{\underline{M}}\;{}_1^2\underline{\underline{C}} + {}_3\underline{\underline{M}}\;{}_3^2\underline{\underline{C}}\;{}_2\underline{\underline{M}}\;{}_2^3\underline{\underline{C}}\;{}_1\underline{\underline{M}}\;{}_1^2\underline{\underline{C}} + {}_1\underline{\underline{M}}\;{}_1^2\underline{\underline{C}}\;{}_2\underline{\underline{M}}\;{}_2^3\underline{\underline{C}}\;{}_3\underline{\underline{M}}\;{}_3^2\underline{\underline{C}} \\
& + {}_1\underline{\underline{M}}\;{}_1^3\underline{\underline{C}}\;{}_3\underline{\underline{M}}\;{}_3^2\underline{\underline{C}})\;{}_2\underline{\underline{M}}\;{}_2^i\underline{a} \quad . \qquad (24)
\end{aligned}
$$

For more spheres, the number of matrix products increases as J^3 with J the total number of spheres.

Once the ${}_1\underline{\underline{S}}$ and ${}_1\underline{B}$ matrices are evaluated, the solution proceeds as above. The effect of the general rotation matrix elements on the magnitude of the $\underline{\underline{C}}$ elements must however be considered before the truncation procedure can be applied. From Appendix C, the rotation addition theorem is given by

$$
R_{n\mu}^{nm}(\alpha,\beta,\gamma) = (-1)^{n-m}\frac{(n-\mu)!}{(n-m)!}\sum_{\sigma}(-1)^{\sigma}\binom{n+m}{n-\mu-\sigma}\binom{n-m}{\sigma}(\sin\beta/2)^{2n}\,.
$$

$$
\cdot\,(\cot\beta/2)^{2\sigma+m+\mu}
$$

The magnitude of $R_{n\mu}^{nm}$ can be readily estimated for particular values of m, μ as

$$
|R_{n\mu}^{nm}| \le |R_{n-n}^{nn}| = (2n)!\,(\sin\beta/2)^{2n} \sim (2n\sin\beta/2)^{2n}\sqrt{4\pi n}\,e^{-2n}
$$

A special case of interest occurs when m or $\mu \sim 0$ since, for this case the $\Omega^{e\mu n}_{e\mu u}$ element is largest.

$$|R^{nn}_{nq}| = \frac{(2n)!}{(n+q)!} (\sin\beta/2)^{n-q} (\cos\beta/2)^{n+q}$$

$$|R^{nm}_{n-n}| = \frac{(2n)!}{(n-m)!} (\sin\beta/2)^{n+m} (\cos\beta/2)^{n-m}$$

$$|R^{n0}_{n-n}| = |R^{nn}_{n0}| = \frac{(2n)!}{n!} (\sin\beta/2)^{n} (\cos\beta/2)^{n}$$

$$\sim \sqrt{2}\,(4n \sin\beta/2)^{n} (\cos\beta/2)^{n} e^{-n} = \frac{1}{\sqrt{\pi n}} (4 \sin\beta/2 \cos\beta/2)^{n}\, n! \quad .$$

The value for both $m = \mu = 0$ gives

$$|R^{n0}_{n0}| = \left| \sum_{\sigma} \binom{n}{n-\sigma} \binom{n}{\sigma} (-1)^{n-\sigma} (\sin\beta/2)^{2(n-\sigma)} (\cos\beta/2)^{2\sigma} \right|$$

$$\sim \left[\frac{n!}{(\frac{n}{2})!\,(\frac{n}{2})!} \right]^{2} (\cos\beta/2)^{n} (\sin\beta/2)^{n} \text{ for n even}$$

$$\sim \left[\frac{n!}{(\frac{n-1}{2})!\,(\frac{n+1}{2})!} \right]^{2} (\cos\beta/2)^{n-1} (\sin\beta/2)^{n} \text{ for n odd.}$$

$$\sim \frac{2}{\pi n} (4 \sin\beta/2 \cos\beta/2)^{n} \quad .$$

The $\underline{\underline{C}}$ elements depend upon both the values of the $\underline{\underline{R}}$ and $\underline{\underline{\Omega}}$ elements. The largest possible products come from the following combination

$$|C^{enn}_{tnn}| \sim |R^{nn}_{n0}|\ |\Omega^{e0n}_{t0n}|\ |R^{n0}_{nn}|$$

$$\sim 2(4n\ \sin\beta/2)^{2n}\ e^{-2n}\ \sqrt{\frac{e}{\pi n}}\ \frac{1}{k_1 d}\left(\frac{4n}{2k_1 d}\right)^{2n}\ (\cos\beta/2)^{2n} \qquad (25)$$

$$\sim \frac{2\sqrt{e}^{-2n}}{k_1 d\ \ \pi n}\left(\frac{16\sin[\beta/2]\cos[\beta/2]}{ek_1 d}\right)^{2n}(n)^{4n} \quad .$$

The dependence of this element on the index n increases much too fast to be compensated by the dependence of $_j\underline{\underline{M}}^e$ on the same index. This means that the use of the truncated matrix for solving the full problem of many spheres using the partial vector wave formalism is in doubt. The study of this problem can be continued on a large-scale computer using two spheres not along the $\vartheta = 0$ axis. The result of truncating the rotation matrix then can be directly compared with a solution that does not require a general rotation operation.

The Eqs. (23) and (24) for these spheres illustrate the difficulty encountered in using the full-wave solution for more than three spheres in a configuration. For more than three spheres the matrix equation is far too cumbersome for use on present computers. A reasonable method of solution must do away with representing each of the possible coordinate transformations with separate matrices. Approximate techniques must be applied to solve the many sphere problems such as described in the next section or by Mathur and Yeh (1964).

The Physical Optics Solution

The physical optics solution to the multiple scattering problem is based on the approximations that the scatterers are far enough apart both for the use of the far-field solution to the single scattering problem and to consider that the scattered wave incident on a second scatterer can be represented as a plane wave. The region of validity of these assumptions can be checked by comparing the physical optics formulation of the problem with the full-wave formulation as developed above. The far-field solution to the single scattering problem is found by examining the asymptotic expansions for the radial functions valid at large values of kr with $k = k_1$ (see Appendix B).

$$
{}^s_j\underset{\sim}{E} \sim \frac{e^{-ikr}}{ikr} \sum_{n=0}^{\infty} i^{n+1} \sum_{m=-n}^{n} \left[\left({}^s_j a^2_{mn} \frac{\partial}{\partial\vartheta} Y_{nm} - {}^s_j a^1_{mn} \frac{mY_{nm}}{\sin\vartheta}\right)\hat{i}_\theta\right.
$$

$$
\left. + \; i\left({}^s_j a^2_{mn} \frac{mY_{nm}}{\sin\vartheta} - {}^s_j a^1_{mn} \frac{\partial}{\partial\vartheta} Y_{nm}\right)\hat{i}_\varphi\right] \quad . \tag{26}
$$

$$
\sim \frac{e^{-ikr}}{ikr} \left(u_\vartheta i_\vartheta + u_\varphi i_\varphi\right) = \frac{e^{-ikr}}{ikr} \underset{\sim}{u}(\hat{r}) \quad .
$$

The representation of the scattered field in cartesian coordinates is given by:

$$\begin{pmatrix} u_k \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} \sin\vartheta\cos\varphi & \cos\vartheta\cos\varphi & -\sin\varphi \\ \sin\vartheta\sin\phi & \cos\vartheta\sin\phi & \cos\varphi \\ \cos\vartheta & -\sin\vartheta & 0 \end{pmatrix} \begin{pmatrix} 0 \\ u_\vartheta \\ u_\varphi \end{pmatrix}$$

$$u_x = \cos\vartheta\cos\varphi\; u_\vartheta - \sin\varphi\; u_\varphi$$

$$= \sum_{n=0}^{\infty} i^{n+1} \sum_{m=-n}^{n} \Bigg\{ {}^s_j a^2_{mn} \left[\cos\vartheta\cos\varphi \frac{\partial}{\partial\vartheta} Y_{nm} - \frac{imY_{nm}}{\sin\vartheta} \sin\varphi \right]$$

$$- {}^s_j a^1_{mn} \left[\cos\vartheta\cos\varphi \frac{mY_{nm}}{\sin\vartheta} - i\sin\varphi \frac{\partial}{\partial\vartheta} Y_{nm} \right] \Bigg\}$$

$$= \sum_{n=0}^{\infty} \frac{i^{n+1}}{2} \sum_{m=-n}^{n} \Bigg\{ \frac{{}^s_j a^2_{mn}}{(2n+1)} \Big[(n-m+1) \big\{ (n+1)(n+m)\; Y_{n-1,\,m-1} + n(n-m+2)\, Y_{n+1,\,m-1} \big\}$$

$$- \big\{ (n+1)\, Y_{n-1,\,m+1} + n\, Y_{n+1,\,m+1} \big\} \Big]$$

$$- {}^s_j a^1_{mn} \Big[(n-m+1)(n+m)\, Y_{n,\,m-1} + Y_{n,\,m+1} \Big] \Bigg\} \quad .$$

$$u_y = \cos\vartheta\sin\varphi\; u_\vartheta + \cos\varphi\; u_\varphi$$

$$= \sum_{n=0}^{\infty} i^{n+1} \sum_{m=-n}^{n} \Bigg\{ {}^s_j a^2_{mn} \left[\cos\vartheta\sin\varphi \frac{\partial}{\partial\vartheta} Y_{nm} + \frac{imY_{nm}}{\sin\vartheta} \cos\varphi \right]$$

$$- {}^s_j a^1_{mn} \left[\cos\vartheta\sin\varphi \frac{mY_{nm}}{\sin\vartheta} + i\cos\varphi \frac{\partial}{\partial\vartheta} Y_{nm} \right] \Bigg\}$$

$$u_y = \sum_{n=0}^{\infty} \left(\frac{i^{n+2}}{2}\right) \sum_{m=-n}^{n} \left\{ \frac{{}^s_j a^2_{mn}}{(2n+1)} \left[(n-m+1) \left\{ (n+1)(n+m) Y_{n-1,m-1} + n(n-m+2) Y_{n+1,m-1} \right\} \right. \right.$$

$$\left. + \left\{ (n+1) Y_{n-1,m+1} + n Y_{n+1,m+1} \right\} \right]$$

$$\left. {}^s_j a^1_{mn} \left[(n-m+1)(n+m) Y_{n,m-1} - Y_{n,m+1} \right] \right\}$$

$$u_z = - \sin\vartheta \, u_\vartheta$$

$$= \sum_{n=0}^{\infty} i^{n-1} \sum_{m=-n}^{n} \left\{ {}^s_j a^2_{mn} \sin\vartheta \frac{\partial}{\partial\vartheta} Y_{nm} - {}^s_j a^1_{mn} \, m Y_{nm} \right\}$$

$$= \sum_{n=0}^{\infty} i^{n+1} \sum_{m=-n}^{n} \left\{ \frac{{}^s_j a^2_{mn}}{(2n+1)} \left[(n+1)(n+m) Y_{n-1,m} - n(n-m+1) Y_{n+1,m} \right] \right.$$

$$\left. - {}^s_j a^1_{mn} \, m Y_{nm} \right\} .$$

For the two-sphere problem the far-field solution to the single scattering problem is evaluated at the center of the first sphere and the Cartesian representation of this field is taken as an incident plane wave on the other sphere. The plane wave amplitude is evaluated by taking $r = d$ and $\vartheta = 0$. For this case, only terms in $Y_{p,0}$ are non-zero.

$$u_x\Big|_{\vartheta=0}=\sum_{n=0}^{\infty}\frac{i^{n+1}}{2}\left[-{}^s_j a^2_{-1n}+n(n+1)\,{}^s_j a^2_{1n}-{}^s_j a^1_{-1n}-n(n+1)\,{}^s_j a^1_{1n}\right]$$

$$u_y\Big|_{\vartheta=0}=\sum_{n=0}^{\infty}\frac{i^{n+2}}{2}\left[{}^s_j a^2_{-1n}+n(n+1)\,{}^s_j a^2_{1n}+{}^s_j a^1_{-1n}-n(n+1)\,{}^s_j a^1_{1n}\right] \tag{27}$$

$$u_z\Big|_{\vartheta=0}=0$$

The wave given above incident on another sphere is represented as

$${}^j_\ell \underset{\sim}{E}=\frac{e^{-ik(d+z)}}{ikd}\left(u_x\Big|_{\vartheta=0}\hat{i}_x+u_y\Big|_{\vartheta=0}\hat{i}_y\right)$$

in the coordinate system of the new sphere. This expression may be expanded in a PSVW series (see Appendix 2) as

$${}^j_\ell \underset{\sim}{E}=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{e=1}^{2}{}^j_\ell a^e_{mn}\,{}_j\underset{\sim}{v}^{e(1)}_{mn}$$

where

$${}^j_\ell a^e_{mn}=\frac{e^{-ikd}\,i^{-(n+1)}(2n+1)}{ikd\,2n(n+1)}\left[-(-1)^e\left(u_x\Big|_{\vartheta=0}-i\,u_y\Big|_{\vartheta=0}\right)\delta_{m,1}\right.$$

$$\left.+\,n(n+1)\left(u_x\Big|_{\vartheta=0}+i\,u_y\Big|_{\vartheta=0}\right)\delta_{m,-1}\right]$$

The incident waves scattered by the first sphere can be expressed using the operator notation as

$$ {}_{\ell}^{j}a_{mn}^{e} = \sum_{p=0}^{\infty} \sum_{q=-p}^{p} \sum_{t=1}^{2} {}_{\ell}^{j}\mathcal{Q}_{tqp}^{emn} \, {}_{j}^{s}a_{qp}^{t} \tag{28} $$

$$ = \sum_{p=0}^{\infty} \frac{i^{p-n} e^{-ikd} (2n+1)}{2\, ikd \; n(n+1)} \left[-\left(p(p+1) \, {}_{j}^{s}a_{1p}^{2} - p(p+1) \, {}_{j}^{s}a_{1p}^{1} \right) \delta_{m,1} (-1)^{e} \right. $$

$$ \left. - n(n+1) \left({}_{j}^{s}a_{-1p}^{2} + {}_{j}^{s}a_{-1p}^{1} \right) \delta_{m,-1} \right] $$

Therefore

$$ {}_{\ell}^{j}\mathcal{Q}_{tqp}^{emn} = -\frac{i^{p-n} e^{-ikd} (2n+1)}{2\, ikd \; n(n+1)} \left[p(p+1) \, \delta_{m,1} \, \delta_{q,1} (-1)^{e+t} + \right. $$

$$ \left. + n(n+1) \, \delta_{m,-1} \, \delta_{q,-1} \right] \tag{29} $$

The coordinate translation matrix $\underline{\underline{\mathcal{Q}}}$ for the physical optics case is far simpler than that for the full wave case, $\underline{\underline{\Omega}}$. The only azimuthal terms that contribute to the multiple scattering solution are for $m = \pm 1$ as required by the assumption of a plane wave incident on the second sphere. The translation theorem in this form is useful for comparison with the full wave case but is still cumbersome for use in the many sphere multiple scattering problem.

The elements of $\underline{\underline{\mathcal{Q}}}$ vary inversely with kd and are nearly independent of n and p for both large. This means that the combination of the $\underline{\underline{\mathcal{Q}}}$ elements with the M elements as required for the multiple scattering problem always yield a matrix equation that can be truncated for solution. This holds true for the general case where rotation operations as well as translation operations are used to generate the multiple scattering equations.

The physical optics solution can be made more useful by reformulating it in terms of plane wave amplitudes. A plane wave incident on sphere j of a multiple sphere configuration as shown in Fig. 1 is represented by its PSVW coefficients

$$ {}^{i}_{j}a^{t}_{qp} = \sum_{\mu=-p}^{p} R^{pq}_{p\mu}(\alpha,\beta,\gamma)\, {}^{0}_{j}a^{t}_{\mu p} $$

where ${}^{0}_{j}a^{t}_{\mu p}$ are the PSVW coefficients for a plane wave traveling along the $\vartheta = 0$ axis.

Using the expression for ${}^{0}_{j}a^{t}_{\mu p}$ given in Appendix B,

$$ {}^{i}_{j}a^{t}_{qp} = \frac{i^{-(p+1)}(2p+1)}{2p(p+1)}\Big[-(-1)^{t}(\mathcal{E}_{\xi} - i\mathcal{E}_{\eta})\, R^{pq}_{p1}(\alpha,\beta,\gamma) $$

$$ + p(p+1)(\mathcal{E}_{\xi} + i\mathcal{E}_{\eta})\, R^{pq}_{p-1}(\alpha,\beta,\gamma)\Big] $$

where (i_ξ, i_η, k^i) is the coordinate system of the incident wave and α, β, γ relate the incident wave coordinate system with a Cartesian system with the z axis along $\vartheta = 0$ as shown in Fig. 1.

The expression for the field scattered by sphere 1 and incident on 2 is given, in this formulation by

$$u_x \Big|_{\vartheta=0} = \sum_{p=0}^{\infty} \frac{i^{p+1}}{2} \left[-{}_jM_p^2 \left({}^i_ja^2_{-1p} - p(p+1)\, {}^i_ja^2_{1p} \right) - {}_jM_p^1 \left({}^i_ja^1_{-1n} + p(p+1)\, {}^i_ja^1_{1n} \right) \right]$$

$$u_y \Big|_{\vartheta=0} = \sum_{p=0}^{\infty} \frac{i^{p+1}}{2} \left[i\, {}_jM_p^2 \left({}^i_ja^2_{-1p} + p(p+1)\, {}^i_ja^2_{1p} \right) + i\, {}_jM_p^1 \left({}^i_ja^1_{-1n} - p(p+1)\, {}^i_ja^1_{1p} \right) \right] .$$

The elements of the rotation matrix used to express $\underset{\sim}{u}$ are required for $q = \pm 1$ only. These elements may be expressed in terms of the Jacobi Polynomial $P_n^{(\alpha,\beta)}(\cos\beta)$ as shown in Edmonds (1957, pg. 58).

$$R^{p1}_{p1}(\alpha,\beta,\gamma) = e^{i(\alpha+\gamma)} \cos^2(\beta/2)\, P^{(0,2)}_{n-1}(\cos\beta)$$

$$R^{p-1}_{p-1}(\alpha,\beta,\gamma) = e^{-i(\alpha+\gamma)} \cos^2(\beta/2)\, P^{(0,2)}_{n-1}(\cos\beta)$$

$$R^{p1}_{p-1}(\alpha,\beta,\gamma) = \frac{1}{p(p+1)}\, e^{i(\alpha-\gamma)} \sin^2(\beta/2)\, P^{(2,0)}_{n-1}(\cos\beta)$$

$$R^{p-1}_{p1}(\alpha,\beta,\gamma) = p(p+1)\, e^{-i(\alpha-\gamma)} \sin^2(\beta/2)\, P^{(2,0)}_{n-1}(\cos\beta) \quad .$$

The expression for u can then be simplified to

$$\underset{\sim}{u}\Big|_{\vartheta=0} = \begin{pmatrix} u_x \\ u_y \\ 0 \end{pmatrix} = \Bigg\{ \cos^2(\beta/2) \left[\sum_{p=0}^{\infty} \frac{2p+1}{2} \left(-{}_jM_p^2 - {}_jM_p^1 \right) P_{p-1}^{(0,2)}(\cos\beta) \right] \cdot$$

$$\cdot \begin{pmatrix} \cos(\alpha+\gamma) & \sin(\alpha+\gamma) & 0 \\ -\sin(\alpha+\gamma) & \cos(\alpha+\gamma) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+ \sin^2(\beta/2) \left[\sum_{p=0}^{\infty} \frac{2p+1}{2} \left({}_jM_p^2 - {}_jM_p^1 \right) P_{p-1}^{(2,0)} \cos\beta) \right] \cdot$$

$$\cdot \begin{pmatrix} \cos(\alpha-\gamma) & -\sin(\alpha-\gamma) & 0 \\ -\sin(\alpha-\gamma) & -\cos(\alpha-\gamma) & 0 \\ 0 & 0 & 0 \end{pmatrix} \Bigg\} \begin{pmatrix} \mathscr{E}_\xi \\ \mathscr{E}_\eta \\ 0 \end{pmatrix} . \qquad (30)$$

This expression can be expressed as

$$\underset{\sim}{u}\Big|_{\vartheta=0} = \underset{=}{A}(\alpha,\beta,\gamma) \cdot \mathscr{E} \qquad (31)$$

where the incident plane wave is given by ${}_j^i\underset{\sim}{E} = {}_j\mathscr{E}\, e^{-i\underset{\sim}{k}^i \cdot \underset{\sim}{r}_j}$

and the scattered wave at the second sphere by ${}_j^s\underset{\sim}{E} = {}_j\underset{=}{A}(\alpha,\beta,\gamma)\, {}_j\underset{\sim}{\mathscr{E}}\, \frac{e^{-ikr_j\cos\vartheta_j}}{ikd}$.

In the coordinate system of the second sphere the incident wave scattered by the first sphere is given by

$$ {}^{i}_{\ell}\underset{\sim}{E} = {}_{j}\underset{=}{A}(\alpha,\beta,\gamma)_{j}\underset{\sim}{\mathcal{E}}^{o} \frac{e^{-ik(r_\ell + d)\cos\vartheta_\ell}}{ikd} $$

which is the first order scattered wave in a system of multiply scattered waves. All higher order scattered waves from sphere 1 incident on sphere 2 would have originally come from sphere 2. This can be represented as

$$ {}^{i}_{j}\underset{\sim}{E} = {}_{j}\underset{=}{A}(0,\pi,0)\ {}_{\ell}\underset{=}{A}(0,\pi,0)\ {}_{\ell}\underset{\sim}{W}\ \frac{e^{-ik(2d+r_\ell\cos\vartheta)}}{(ikd)^2} $$

where ${}_{\ell}\underset{\sim}{W}$ is the plane wave amplitude originally incident on sphere 2 from the direction of sphere 1.

The multiple scattering problem can be solved directly by computing the incident field for many higher ordered scatterings. This system eventually terminates because each scattering returns less energy to the other sphere. The problem can also be formulated in a "self consistant" manner using undetermined plane wave amplitudes for the path connecting the two spheres. As was done for the PSVW matrices above. The solution can be formulated by considering the plane waves incident on each sphere. If, $\underset{\sim}{W}$ denotes the plane wave amplitude of the plane wave incident on sphere 1

$$ {}_{1}\underset{\sim}{W} = \left[\left({}_{2}\underset{=}{A}(\alpha,\pi-\beta,\gamma)_{2}\underset{\sim}{\mathcal{E}}^{o} + {}_{2}\underset{=}{A}(0,\pi,0)_{2}W \right) \right] \frac{e^{ikd}}{ikd} \tag{32} $$

$$ {}_{2}\underset{\sim}{W} = \left[\left({}_{1}\underset{=}{A}(\alpha,\beta,\gamma)_{1}\underset{\sim}{\mathcal{E}}^{o} + {}_{1}\underset{=}{A}(0,\pi,0)_{1}W \right) \right] \frac{e^{ikd}}{ikd} $$

which is analogous to Eq. (12) for two spheres. This has as its solution with

$$\underset{=}{F} = \underset{=}{A} \frac{e^{ikd}}{ikd}$$

$$ {}_1\underset{\sim}{W} = {}_2\underset{=}{F}(\alpha, \pi-\beta, \gamma)\, {}_2\underset{\sim}{\mathscr{E}} + {}_2\underset{=}{F}(0, \pi, 0)\left[{}_1\underset{=}{F}(\alpha, \beta, \gamma)\, {}_1\underset{\sim}{\mathscr{E}} + {}_1\underset{=}{F}(0, \pi, 0)\, {}_1\underset{\sim}{W} \right] $$

$$ \left[\underset{=}{1} - {}_2\underset{=}{F}(0, \pi, 0)\, {}_1\underset{=}{F}(0, \pi, 0) \right] {}_1\underset{\sim}{W} = {}_2F(\alpha, \pi-\beta, \gamma)\, {}_2\underset{\sim}{\mathscr{E}} + {}_2\underset{=}{F}(0, \pi, 0)\, {}_1\underset{=}{F}(\alpha, \beta, \gamma)\, {}_1\underset{\sim}{\mathscr{E}} \,. $$

For the incident plane wave, the plane wave amplitudes at each sphere are related by

$$ {}_2\underset{\sim}{\mathscr{E}} = {}_1\underset{\sim}{\mathscr{E}}\, e^{-i\underset{\sim}{k}^{i} \cdot \underset{\sim}{d}} $$

The resultant equation for ${}_1\underset{\sim}{W}$ is given by

$$ \left[\underset{=}{1} - {}_2\underset{=}{F}(0, \pi, 0) \right] {}_1\underset{\sim}{W} = \left[{}_2F(\alpha, \pi-\beta, \gamma)\, e^{-ik^{i}d} + {}_2\underset{=}{F}(0, \pi, 0)\, {}_1\underset{=}{F}(\alpha, \beta, \gamma) \right] {}_1\underset{\sim}{\mathscr{E}} \qquad (33) $$

which expresses a multiple scattering equation analogous to Eq. (14) above but with only three dimensional vectors and matrices. The total solution to the problem then is found using the solution $\underset{=}{W}$ together with the single scattering amplitude $\underset{=}{A}$ for the angles given on Fig. 1 with the observation point at $(r_j, \vartheta_j, \varphi_j)$.

$$ {}^{s}\underset{\sim}{E} = \left[{}_2\underset{=}{A}(\psi_2, \Theta_2, \delta_1)\ {}_2\underset{\sim}{\mathscr{E}} + {}_2\underset{=}{A}(\varphi_2, \vartheta_2, 0)\ {}_2\underset{\sim}{W} \right] \frac{e^{-ikr_2}}{ikr_2} $$

$$ + \left[{}_1\underset{=}{A}(\psi_1, \Theta_1, \delta_2)\ {}_1\underset{\sim}{\mathscr{E}} + {}_1\underset{=}{A}(\varphi_1, \pi - \vartheta_1, 0)\ {}_1\underset{\sim}{W} \right] \frac{e^{-ikr_1}}{ikr_1} \quad . $$

where the angles depicted in the figure and required for the computations are

$$ \psi_j = \cos^{-1}\left[\hat{i}_\eta \cdot (\hat{r}_j \times \hat{k}_i) \right] $$

$$ \Theta_j = \cos^{-1}\left[\hat{k}_i \cdot \hat{r}_j \right] $$

$$ \delta_j = \cos^{-1}\left[(\hat{k}_i \times \hat{r}_j) \cdot (\hat{i}_z \times \hat{r}_j) \right] $$

$$ \alpha_j = \cos^{-1}\left[\hat{i}_\eta \cdot (\hat{i}_z \times \hat{k}_i) \right] $$

$$ \beta_j = \cos^{-1}\left[\hat{k}_i \cdot \hat{i}_z \right] $$

$$ \gamma_j = \cos^{-1}\left[\hat{i}_y \cdot (\hat{i}_z \times \hat{k}_i) \right] $$

$$ \vartheta_j = \cos^{-1}\left[\hat{i}_z \cdot \hat{r}_j \right] $$

$$\varphi_j = \cos^{-1}\left[\hat{i}_y \cdot (\hat{i}_z \times \hat{r}_j)\right]$$

$$\hat{i}_z = \hat{d}$$

and $\hat{i}_y$ is arbitrary

Conclusions

The full-wave and physical-optics solution to the two and three sphere multiple scattering problem as derived above are amenable to solution using a large scale computer. The full-wave solution can be used for up to three spheres whose single scattering solution lies in the Mie range. For more than three spheres, this solution becomes extremely complicated. The physical optics solution suffers from the same problems of complication for more than three spheres if used in the PSVW coefficient matrix form given above. This solution can however be cast in a simpler form that can be used for more spheres as was done above.

The full-wave solution requires a consideration of all the azimuthal index terms in the series for a general multiple scattering problem. The validity of the truncated series solution is in doubt in the general case. To ascertain the validity of such a solution, the problem must be considered using the computer. For the two-sphere problem, a solution using the full-wave treatment can always be obtained.

The full-wave solution can be programmed to provide a check on the simpler approximate solutions to the multiple scattering problem. A check on the full-wave solution can also be found in the body of existing experimental results for fixed configurations of two and three spheres. The physical optics solution generated above has the advantages of directly using the single scattering solution to the plane-wave problem and in providing an operator equation that relates incident and scattered plane-wave amplitudes. The details of this approximate solution can be varied by using the full near-field expression close to the scatterer or by taking the scattered wave from one sphere incident on the other as a series or integral function of plane waves propagating either in different directions or at different velocities. These more complicated approximate solutions may be valid over more of the space available to the scatterers and provide a better basis for solving the many sphere multiple scattering problem. Each of these solutions can be prepared for the two and three-sphere configurations for comparison with the full-wave solution. The usefulness of the full-wave solution is not in providing a basis for the solution of more complicated multiple scattering problems, but in providing a yardstick against which approximate solutions can be compared.

APPENDIX A

NOTATION

The partial spherical vector wave (PSVW) functions and their coefficients can be expressed in a compact form using the following notation.

PSVW function: $\,_{j}\underset{\sim}{v}^{e(\kappa)}_{mn\ell}$

where e refers to the vector wave function with e = 1 specifying $\underset{\sim}{m}$ and e = 2 specifying $\underset{\sim}{n}$ (Stratton notation)

κ refers to the radial function used with κ = 1 specifying the spherical Bessel function and κ = 4 specifying the spherical Hankel function of the second kind.

j refers to the coordinate system centered on sphere j.

m refers to the azimuthal index of the spherical harmonic function

$Y_{nm}(\vartheta, \varphi) = P_n^m(\cos\vartheta)\, e^{im\varphi}; \qquad m - n \le m \le n.$

n refers to the polar index of the spherical harmonic function.

ℓ refers to the wave number to be used in computing the wave function.

PSVW coefficient: $\,^{s}_{j}a^{e}_{mn}$

where j, e, m, n have the same meaning as above

s refers to the part of the electromagnetic field represented by the product of the coefficient and the partial vector wave function. The letter s is used for a scattered wave, i for an incident wave, t for a transmitted wave, and 0 for a plane wave directed along the ϑ = 0 axis.

With this notation, the position of the index is important. When the particular index is not important as with a discussion about $\underset{\sim}{v}$ for any sphere, the index is omitted.

Matrix operator element: ${}^{\ell}_{j}\mathscr{Q}^{emn}_{uqp}$

where j, e, m, n refer to one set of PSVW coefficients with the meanings above and ℓ, u, q, p refer to a second set of PSVW coefficients. The matrix operator is used to transform one set of coefficients to another as for the coordinate transformation formulas.

APPENDIX B

PROPERTIES OF PSVW FUNCTIONS

Vector fields with zero divergence may be expanded in two types of partial spherical vector waves, the m and n waves of Stratton.

Let $\quad Y_{nm}(\hat{\underset{\sim}{r}}) = Y_{nm}(\vartheta, \varphi) = P_n^m(\cos\vartheta)\, e^{im\varphi} \qquad -n \le m \le n; \quad 0 \le n \le \infty$

$$\psi_{nm}(\underset{\sim}{r}) = \psi_{nm}^{(\kappa)}(r, \vartheta, \varphi) = z_n^{(\kappa)}(kr)\, Y_{nm}$$

where $\quad z_n^{(1)} = j_n, \; z_n^{(2)} = n_n, \; z_n^{(3)} = h_n^{(1)}$, and $z_n^{(4)} = h_n^{(2)}$

are spherical Bessel functions

Then the partial spherical vector waves are defined as

$$\underset{\sim}{v}_{mn}^{1(\kappa)} = \underset{\sim}{m}_{mn}^{(\kappa)} = \nabla \times [\psi(\underset{\sim}{r})\underset{\sim}{r}] = z_n^{(\kappa)}(kr)\left[\frac{im\, Y_{nm}(r)}{\sin\vartheta}\,\hat{i}_{\vartheta} - \frac{\partial}{\partial\vartheta} Y_{nm}(\hat{r})\,\hat{i}_{\varphi}\right] \tag{B-1}$$

$$\underset{\sim}{v}_{mn}^{2(\kappa)} = \underset{\sim}{n}_{mn}^{(\kappa)} = \frac{1}{k}\nabla \times \nabla \times [\psi(\underset{\sim}{v})\underset{\sim}{r}] = \frac{n(n+1)}{kr}\, z_n^{(\kappa)}(kr)\; Y_{nm}(\hat{r})\,\hat{i}_r$$

$$+ \frac{1}{kr}\left(\frac{\partial}{\partial r}\left[r z_n^{(\kappa)}(kr)\right]\right)\left[\frac{\partial}{\partial\vartheta} Y_{nm}(\hat{r})\,\hat{i}_{\vartheta} + \frac{im\, Y_{nm}(r)}{\sin\vartheta}\,\hat{i}_{\varphi}\right] \tag{B-2}$$

The PSVW functions have the following orthogonality properties

$$\left(\underset{\sim}{v}_{mn}^{1(\kappa)}, \underset{\sim}{v}_{\mu\nu}^{1(\sigma)}\right) = \int_0^{2\pi}\int_0^{\pi} \underset{\sim}{v}_{mn}^{1(\kappa)*} \cdot \underset{\sim}{v}_{\mu\nu}^{1(\sigma)} \sin\vartheta \, d\vartheta \, d\varphi$$

$$= \frac{4\pi n(n+1)(n+m)!}{(2n+1)(n-m)!} \, z_n^{(\kappa)*}(kr) \, z_n^{(\sigma)}(kr) \, \delta_{m\mu} \, \delta_{n\nu} \quad . \tag{B-3}$$

$$\left(\underset{\sim}{v}_{mn}^{1(\kappa)}, \underset{\sim}{v}_{\mu\nu}^{2(\sigma)}\right) = \left(\underset{\sim}{v}_{mn}^{2(\kappa)}, \underset{\sim}{v}_{\mu\nu}^{1(\sigma)}\right) = 0 \tag{B-4}$$

$$\left(\underset{\sim}{v}_{mn}^{2(\kappa)}, \underset{\sim}{v}_{\mu\nu}^{1(\sigma)}\right) = \frac{4\pi n(n+1)(n+m)!}{(2n+1)^2(n-m)!}\left[(n+1)\, z_{n-1}^{(\kappa)*} \, z_{n-1}^{(\sigma)} + n z_{n+1}^{(\kappa)*} \, z_{n+1}^{(\sigma)}\right] \delta_{m\mu} \, \delta_{n\nu} \quad . \tag{B-5}$$

The asymptotic properties of the PSVW functions depend upon the form of the radial function used. For scattered waves, the spherical Hankel function of the second kind is used to represent spherical outward traveling waves

$$z_n^{(4)}(kr) = h_n^{(2)}(kr) \sim -i^n \frac{e^{-ikr}}{ikr} + i^{n+2} \frac{e^{-ikr}}{ikr}$$

$$\frac{1}{kr}\frac{\partial}{\partial r}\left[r z_n^{(4)}(kr)\right] = (n+1)\frac{z_n^{(4)}(kr)}{kr} - z_{n+1}^{(4)}(kr) \sim i^{n+1} \frac{e^{-ikr}}{ikr}$$

$$\underset{\sim}{v}_{mn}^{1(4)} \sim i^{n+2} \frac{e^{-ikr}}{ikr}\left[\frac{im\, Y_{nm}(r)}{\sin\vartheta}\, i_{\vartheta} - \frac{\partial}{\partial\vartheta} Y_{nm}(r)\, i_{\varphi}\right] \tag{B-6}$$

$$\underset{\sim}{v}_{mn}^{2(4)} \sim i^{n+1}\,\frac{e^{-ikr}}{ikr}\left[\frac{\partial}{\partial\vartheta} Y_{nm}(r)\,\hat{i}_{\vartheta} + \frac{im\,Y_{nm}(r)}{\sin\vartheta}\,\hat{i}_{\varphi}\right] \tag{B-7}$$

The expansion of a plane wave incident along the $\vartheta = 0$ direction is expressed as

$$^{i}\underset{\sim}{E} = \underset{\sim}{\mathcal{E}}\, e^{-i\underset{\sim}{k}^{i}\cdot\underset{\sim}{r}} = \left(\mathcal{E}_x \hat{i}_x + \mathcal{E}_y \hat{i}_y\right) e^{-ikz}$$

$$= \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{t=1}^{2} {}^{0}a_{mn}^{t}\,\underset{\sim}{v}_{mn}^{t(1)}$$

$${}^{0}a_{mn}^{t} = \frac{\left(\underset{\sim}{v}_{mn}^{t(1)},\ \underset{\sim}{\mathcal{E}}\,e^{-ik}\right)}{\left(\underset{\sim}{v}_{mn}^{t(1)},\ \underset{\sim}{v}_{mn}^{t(1)}\right)}$$

$$\therefore\ {}^{0}a_{mn}^{1}\left(\underset{\sim}{v}_{mn}^{1(1)},\ \underset{\sim}{v}_{mn}^{1(1)}\right) = \int_0^{2\pi}\int_0^{\pi} j_n(kr)\left[-\frac{im\,Y^{*}_{nm}}{\sin\vartheta}\,\hat{i}_{\vartheta} - \frac{\partial}{\partial\vartheta} Y^{*}_{nm}\,\hat{i}_{y}\right]\cdot$$

$$\cdot\left[(\mathcal{E}_x\cos\varphi + \mathcal{E}_y\sin\varphi)\,(\hat{i}_{\vartheta}\sin\vartheta + \hat{i}_{\vartheta}\cos\vartheta)\right.$$

$$\left. - (\mathcal{E}_x\sin\varphi - \mathcal{E}_y\cos\varphi)\,\hat{i}_{y}\right] e^{-ikr\cos\vartheta}\sin\vartheta\, d\vartheta\, d\varphi$$

now $$e^{-ikr\cos\vartheta} = \sum_{\nu=0}^{\infty} i^{-\nu}\, j_{\nu}(kr)\, P_{\nu}(\cos\vartheta)\,(2\nu+1)$$

$$\therefore\ {}^0a^1_{mn}\left(\underset{\sim}{v}^{1(1)}_{mn},\ \underset{\sim}{v}^{1(1)}_{mn}\right) = -\,2\pi n(n+1)\,i^{-n+1}\ j^2_n(kr)\left[\left(\mathscr{E}_x - i\,\mathscr{E}_y\right)\delta_{m,1} + \frac{\left(\mathscr{E}_x + i\,\mathscr{E}_y\right)\delta_{m,-1}}{n(n+1)}\right]$$

$${}^0a^1_{mn} = \frac{i^{-(n+1)}(2n+1)}{2n(n+1)}\left[\left(\mathscr{E}_x - i\,\mathscr{E}_y\right)\delta_{m,1} + n(n+1)\left(\mathscr{E}_x + i\,\mathscr{E}_y\right)\delta_{m,-1}\right] .$$

In a similar manner

$${}^0a^2_{mn} = \frac{\left(\underset{\sim}{v}^{2(1)}_{mn},\ \mathscr{E}e^{-ikz}\right)}{\left(\underset{\sim}{v}^{2(1)}_{mn},\ v^{2(1)}_{mn}\right)} = \frac{i^{-(n+1)}(2n+1)}{2n(n+1)}\left[-\left(\mathscr{E}_x - i\,\mathscr{E}_y\right)\delta_{m,1} + n(n+1)\cdot\right.$$

$$\left.\cdot\left(\mathscr{E}_x + i\,\mathscr{E}_y\right)\delta_{m,-1}\right] .$$

Hence

$${}^0a^t_{mn} = \frac{i^{-(n+1)}(2n+1)}{2n(n+1)}\left[-(-1)^t\left(\mathscr{E}_x - i\,\mathscr{E}_y\right)\delta_{m,1} + n(n+1)\left(\mathscr{E}_x + i\,\mathscr{E}_y\right)\delta_{m,-1}\right] . \quad \text{(B-8)}$$

APPENDIX C

PSVW ADDITION THEOREMS

The general addition theorems required for a combined translation and rotation operation are given by Curzan (1962) and Stein (1961). The general addition theorem may be decomposed into two special operations, one a rotation of coordinates and the second a translation of coordinate system center along the special, $\vartheta = 0$ axis. Both addition theorems use as a start the addition theorems for scalar spherical harmonics. The rotation addition theorem for scalar spherical harmonics is given by Edmonds (1957) as

$$Y_{nm}(\vartheta, \varphi) = \sum_{\mu=-n}^{n} (-1)^{\mu-m} D_{\mu m}^{(n)}(\alpha, \beta, \gamma)\left[\frac{(n-\mu)!\,(n+m)!}{(n+\mu)!\,(n-m)!}\right]^{1/2} Y_{n\mu}(\vartheta', \varphi')$$

where $D_{\mu m}^{(n)}(\alpha, \beta, \gamma) = e^{i\mu\alpha}\, d_{\mu m}^{(n)}(\beta)\, e^{im\gamma}$

$$d_{\mu m}^{(n)}(\beta) = \left[\frac{(n+\mu)!\,(n-\mu)!}{(n+m)!\,(n-m)!}\right]^{1/2} \sum_{\sigma} \binom{n+m}{n-\mu-\sigma}\binom{n-m}{\sigma} (-1)^{m-\mu-\sigma} \,.$$

$$\cdot\, [\cos(\beta/2)]^{2\sigma+\mu+m}\, [\sin(\beta/2)]^{2n-2\sigma-\mu-m}$$

α, β, γ are the Euler angles, describing the rotation as shown in Fig. 1.

$Y_{nm}(\vartheta, \varphi) = P_n^m(\cos\vartheta)\, e^{im\varphi}$ are unnormalized spherical harmonics.

Using this additional theorem

$$\psi_{nm}^{(\kappa)}(r, \vartheta, \varphi) = z_n^{(\kappa)}(kr)\ Y_{nm}(\vartheta, \varphi) = z_n^{(\kappa)}(kr) \sum_{\mu=-n}^{n} R_{n\mu}^{nm}(\alpha, \beta, \gamma)\, Y_{n\mu}(\vartheta', \varphi')$$

$$= \sum_{p=0}^{\infty} \sum_{q=-p}^{p} R_{pq}^{nm}(\alpha, \beta, \gamma)\, \psi_{pq}^{(\mu)}(r, \vartheta', \varphi') \tag{C-9}$$

where

$$R_{pq}^{nm}(\alpha, \beta, \gamma) = \delta_{np} (-1)^{q-m} \left[\frac{(n-q)!\,(n+m)!}{(n+q)!\,(n-m)!}\right]^{1/2} e^{iq\alpha}\, d_{qm}^{(n)}(\beta)\, e^{im\gamma}$$

$$= \left[\frac{(n-q)!}{(n-m)!}\, e^{i(q\alpha+m\gamma)} \sum_{\sigma} (-1)^{\sigma} \binom{n+m}{n-q-\sigma}\binom{n-m}{\sigma} [\cos(\beta/2)]^{2\sigma+q+m} \cdot\right.$$

$$\left. \cdot\ [\sin(\beta/2)]^{2n-2\sigma-q-m}\right] \delta_{np}$$

and since

$$\underset{\sim}{v}_{mn}^{1(\mu)}(r, \vartheta, \varphi) = \nabla \times \left(\psi_{nm}^{(\mu)}(r, \vartheta, \varphi)\underset{\sim}{r}\right)$$

$$= \nabla' \times \left(\sum_{p=0}^{\infty} \sum_{q=-p}^{p} R_{pq}^{nm}(\alpha, \beta, \gamma)\, \psi_{pq}^{(\mu)}(r, \vartheta', \varphi')\, \underset{\sim}{r}'\right)$$

$$= \sum_{p=0}^{\infty} \sum_{q=-p}^{p} R_{pq}^{nm}(\alpha, \beta, \gamma)\, \underset{\sim}{v}_{pq}^{1(\mu)}(r, \vartheta', \varphi')$$

$$\underset{\sim}{v}^{2(\mu)}_{mn}(r,\vartheta,\varphi) = \nabla' \times \nabla' \times \left(\sum_{p=0}^{\infty} \sum_{q=-p}^{p} R^{nm}_{pq}(\alpha,\beta,\gamma)\, \psi^{(\mu)}_{pq}(\underset{\sim}{r},\vartheta',\varphi')\, \underset{\sim}{r}' \right)$$

$$= \sum_{p=0}^{\infty} \sum_{q=-p}^{p} R^{nm}_{pq}(\alpha,\beta,\gamma)\, \underset{\sim}{v}^{2(\mu)}_{pq}(r,\vartheta',\varphi') \quad .$$

Therefore the rotation transformation is given in full as

$$\underset{\sim}{v}^{t(\mu)}_{mn}(r,\vartheta,\varphi) = \sum_{p=0}^{\infty} \sum_{q=-p}^{p} \sum_{e=1}^{2} R^{tnm}_{epq}(\alpha,\beta,\gamma)\, \underset{\sim}{v}^{e(\mu)}_{pq}(r,\vartheta',\varphi') \tag{C-10}$$

where $\quad R^{tnm}_{epq} = \delta_{te}\, R^{nm}_{pq}(\alpha,\beta,\gamma) \quad .$

A special case of the rotation matrix is useful in problems with a translation axis along $\vartheta = \pi$.

$$R^{nm}_{pq}(0,\pi,0) = (-1)^{q-m} \left[\frac{(n-q)!\,(n+m)!}{(n+q)!\,(n-m)!} \right]^{1/2} d^{(n)}_{qm}(\pi)\, \delta_{np}$$

$$= (-1)^{n-m} \left[\frac{(n+m)!\,(n+m)!}{(n-m)!\,(n-m)!} \right]^{1/2} \delta_{q-m}\, \delta_{np}$$

$$= (-1)^{n-m} \frac{(n+m)!}{(n-m)!}\, \delta_{m-q}\, \delta_{np} \quad .$$

The rotation of coordinate system transformation affects only the azimuthal index m of $\underset{\sim}{v}^{t(\mu)}_{nm}$.

The translation addition theorem for scalar spherical harmonics is given by Friedman and Russek (1954) as corrected by Cruzan

$$\psi^{(\kappa)}_{nm}(\underset{\sim}{r}) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \sum_{p} (-1)^{\mu}\, i^{\nu+p-n} (2\nu + 1)\, a(m,n\,|-\mu,\nu\,|p)\, z^{(\kappa)}_{p}(kd) \cdot$$

$$\cdot\, P^{m-\mu}_{p}(\cos\vartheta_o)\, e^{i(m-\mu)\vartheta_o}\, \psi^{(1)}_{\nu\mu}(\underset{\sim}{r}') \qquad \text{(C-11)}$$

$$r' \leq d$$

and

$$\psi^{(\kappa)}_{nm}(\underset{\sim}{r}) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \sum_{p} (-1)^{\mu}\, i^{\nu+p-n} (2\nu + 1)\, a(m,n\,|-\mu,\nu\,|p)\, j_{\nu}(kd) \cdot$$

$$\cdot\, P^{\mu}_{\nu}(\cos\vartheta_o)\, e^{i\mu\vartheta_o}\, \psi^{(\kappa)}_{pm-\mu}(\underset{\sim}{r}') \quad . \qquad \text{(C-12)}$$

$$r' \geq d$$

where

$$a(m,n\,|-\mu,\nu\,|p) = (-1)^{m-\mu}(2p + 1)\left[\frac{(n+m)!\,(\nu-\mu)!\,(p-m+\mu)!}{(n-m)!\,(\nu+\mu)!\,(p+m-\mu)!}\right]^{1/2} \cdot$$

$$\cdot \begin{pmatrix} n & \nu & p \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} n & \nu & p \\ m & -\mu & -m+\mu \end{pmatrix}$$

$$\begin{pmatrix} n & \nu & p \\ m & \mu & q \end{pmatrix} = \frac{(-1)^{n-\nu-\mu}}{\sqrt{2\nu+1}} \, (nm\nu\mu \,|\, n\nu pq) = \text{Wigner 3-j symbol (see Edmonds)}$$

$$= \delta_{m+\mu,\,-q} \left[\frac{(2p+1)\,(p+n-\nu)!\,(p+\nu-n)!\,(n+\nu-p)!}{(2\nu+1)\,(p+n+\nu+1)!} \right]^{1/2} .$$

$$\cdot \left[(n+m)!\,(n-m)!\,(\nu+\mu)!\,(\nu-\mu)!\,(p+q)!\,(p-q)! \right]^{1/2} \quad .$$

$$\sum_{\sigma=0}^{p+|q|} \frac{(-1)^{\sigma+n-\nu-\mu}}{\sigma!} \left[(n+\nu-p-\sigma)!\,(n-m-\sigma)!\,(\nu+\mu-\sigma)!\,(p-\nu+m+\sigma)!\,(p-n-m+\sigma)! \right]^{-1}$$

$$\neq 0 \text{ if } |n-v| \leq p \leq n+v \quad .$$

For the simple case of translation along $\vartheta_o = 0$,

$$\psi_{nm}^{(\kappa)}(\underset{\sim}{r}) = \sum_{\nu=0}^{\infty} \sum_{p} (-1)^m i^{\nu+p-n} (2\nu + 1) \, a(m, n \,|\, -m, \nu \,|\, p) \, z_p^{(\kappa)}(kd) \, \psi_{\nu m}^{(1)}(\underset{\sim}{r}')$$

$$r' \leq d \quad .$$

$$= \sum_{\nu=0}^{\infty} T_{\nu n}^{(m,\kappa)}(d) \, \psi_{\nu m}^{(1)}(\underset{\sim}{r}') \qquad r' \leq d$$

$$= \sum_{\nu=0}^{\infty} \sum_{p} (-1)^{\nu} i^{\nu+p-n} (2\nu + 1) \, a(m, n \,|\, 0, \nu \,|\, p) \, j_{\nu}(kd) \, \psi_{pm}^{(\kappa)}(\underset{\sim}{r}')$$

$$r' \geq d \quad .$$

$$= \sum_{p}^{\infty} U_{pn}^{(m)}(d) \, \psi_{pm}^{(\kappa)}(\underset{\sim}{r}') \qquad r' \geq d$$

The evaluation of $T_{\nu n}^{(m,\kappa)}(d)$ starts with the properties of the Wigner 3-j symbols

$$\begin{pmatrix} n & \nu & p \\ m & -m & 0 \end{pmatrix} = \begin{pmatrix} p & n & \nu \\ 0 & m & -m \end{pmatrix} = (-1)^{p-n-m}(2n+1)^{-1/2}(p\,0\,n\,m \mid p\,n\,\nu\,m)$$

$$= (-1)^{p-n-m}\left[\frac{(2\nu+1)\,(p+n-\nu)!\,(p-n+\nu)!\,(-p+n+\nu)!}{(2n+1)\qquad (p+n+\nu+1)!}\right]^{1/2} \cdot$$

$$\cdot \sum_s \frac{(-1)^s p!\,[(n+m)!\,(n-m)!\,(\nu+m)!\,(\nu-m)!]^{1/2}}{s!\,(p+n-\nu-s)!\,(p-s)!\,(n+m-s)!\,(\nu-n+s)!\,(\nu-p-m+s)!}$$

$$= (-1)^{p-n-m}\left[\frac{(2\nu+1)(p+n-\nu)!\,(p-n+\nu)!\,(-p+n+\nu)!}{(2n+1)\qquad (p+n+\nu+1)!}\right]^{1/2}\left[\frac{(n-m)!\,(\mu+m)!}{(n+m)!\,(\mu-m)!}\right]^{1/2} \cdot$$

$$\cdot \sum_s (-1)^s \binom{n+m}{s}\binom{\nu-m}{p-s}\binom{p}{\nu-n+s}$$

And $\begin{pmatrix} n & \nu & p \\ 0 & 0 & 0 \end{pmatrix} = (-1)^{p-n}\left[\frac{2\nu+1}{2n+1}\,\frac{(p+n-\nu)!\,(p-n+\nu)!\,(-p+n+\nu)!}{(p+n+\nu+1)!}\right]\sum_s (-1)^s\binom{n}{s}\binom{\nu}{p-s}\binom{p}{\nu-n+s}$.

Therefore

$$a(m,n \mid -m,\nu \mid p) = (-1)^m\,\frac{(2p+1)\,(2\nu+1)}{(2n+1)}\left[\frac{(p+n-\nu)!\,(p-n+\nu)!\,(p+n+\nu)!}{(p+n+\nu+1)!}\right] \cdot$$

$$\cdot\left[\sum_s (-1)^s\binom{n}{s}\binom{\nu}{p-s}\binom{p}{\nu-n+s}\quad \sum_\sigma (-1)^\sigma\binom{n+m}{\sigma}\binom{\nu-m}{p-\sigma}\binom{p}{\nu-n+\sigma}\right]$$

$$a(m,n\,|-m,\nu\,|p) = (-1)^m G^n_{p\nu}\, H^{n,m}_{p\nu}\, H^{n,0}_{p\nu} \tag{C-13}$$

where

$$G^n_{p\nu} = \frac{(2p+1)(2\nu+1)}{(2n+1)}\left[\frac{(p+n-\nu)!\,(p-n+\nu)!\,(-p+n+\nu)!}{(p+n+\nu+1)!}\right]$$

$$H^{n,m}_{p\nu} = \sum_s (-1)^s \binom{n+m}{s}\binom{\nu-m}{p-s}\binom{p}{\nu-n+s} \;.$$

Then

$$T^{(m,\kappa)}_{\nu n}(d) = \sum_p G^n_{p\nu}\, H^{n,m}_{p\nu}\, H^{n0}_{p\nu}\, z^{(\kappa)}_p(kd)\, i^{\nu+p-n} \;. \tag{C-14}$$

In this case, $U^m_{pn}(d)$ is found from $T^{(m,\kappa)}_{\nu n}(d)$ by renaming p and ν due to the symmetries of $a(m,n\,|-\mu,\nu\,|p)$ as

$$U^{(m)}_{pn}(d) = T^{(m,1)}_{pn}(d) \;. \tag{C-15}$$

The PSVW addition theorems are found using the definition of $\underset{\sim}{v}^{t(\kappa)}_{mn}$ as

$$\underset{\sim}{v}^{1(\kappa)}_{mn} = \nabla' \times \left[(\underset{\sim}{d} + \underset{\sim}{r}') \sum_{\nu=0} T^{(m,\kappa)}_{\nu n}\, \psi^{(1)}_{\nu m}(\underset{\sim}{r}')\right]$$

$$\therefore \quad {}_{j}\underset{\sim}{v}{}^{1(\kappa)}_{mn} = \nabla' \times \left[\underset{\sim}{r}' \sum_{\nu=0}^{\infty} T^{(m,\kappa)}_{\nu m} \psi^{(m,1)}_{\nu m} (\underset{\sim}{r}') \right] +$$

$$+ \nabla' \times \left[\underset{\sim}{d} \sum_{\nu=0}^{\infty} T^{(m,\kappa)}_{\nu n} \psi^{(1)}_{\nu m} (\underset{\sim}{r}') \right]$$

from Stein, the expansion for the second term on the right-hand side is given as

$$\sum_{\nu=0}^{\infty} T^{(m,\kappa)}_{\nu n} \nabla' \times \left[\underset{\sim}{d}\, {}_{e}\psi^{(1)}_{\nu m} (\underset{\sim}{r}') \right] = \sum_{\nu=0}^{\infty} T^{(m,\kappa)}_{\nu n} \left[\frac{kdim}{\nu(\nu+1)}\, {}_{\ell}\underset{\sim}{v}{}^{2(1)}_{\nu m} + \right.$$

$$\left. + \frac{kd}{2\nu+1} \left(\frac{\nu+m}{\nu}\, {}_{\ell}\underset{\sim}{v}{}^{1(1)}_{\nu-1,m} + \frac{\nu-m+1}{\nu+1}\, {}_{\ell}\underset{\sim}{v}{}^{1(1)}_{\nu+1,m} \right) \right]$$

$$= \sum_{\nu=0}^{\infty} \frac{T^{(m,\kappa)}_{\nu n} kdim}{\nu(\nu+1)}\, {}_{\ell}\underset{\sim}{v}{}^{2(1)}_{\nu m} + \left[\frac{(kd)\, T^{(m,\kappa)}_{\nu+1 n} (\nu+m+1)}{(2\nu+3)(\nu+1)} + \frac{kd\, T^{(m,\kappa)}_{\nu-1 n} (\nu-m)}{(2\nu-1)} \right] {}_{\ell}\underset{\sim}{v}{}^{1(1)}_{\nu m}$$

$$\therefore \; {}_{j}\underset{\sim}{v}{}^{1(\kappa)}_{mn} = \sum_{\nu=0}^{\infty} \left[T^{(m,\kappa)}_{\nu n} + \frac{kd(\nu+m+1)}{(2\nu+3)(\nu+1)} T^{(m,\kappa)}_{\nu+1 n} + \frac{kd(\nu-m)}{\nu(2\nu-1)} T^{(m,\kappa)}_{\nu-1 n} \right] {}_{\ell}\underset{\sim}{v}{}^{1(1)}_{\nu m} +$$

$$+ \frac{ikd\, T^{(m,\kappa)}_{\nu n}}{\nu(\nu+1)}\, {}_{\ell}\underset{\sim}{v}{}^{2(1)}_{\nu m}$$

$$= \sum_{p=0}^{\infty} \sum_{q=-p}^{p} \sum_{e=1}^{2} {}^{\ell}_{j}\Omega^{mn(\kappa)}_{eqp}(kd)\, {}_{\ell}\underset{\sim}{v}{}^{e(1)}_{qp} \qquad \text{(C-16)}$$

where

$$ {}^{e}_{j}\Omega^{1mn(\kappa)}_{1qp}(kd) = \left[T^{(m,\kappa)}_{pn} + \frac{kd\,(p+m+1)}{(2p+3)\,(p+1)}\, T^{(m,\kappa)}_{p+1\,n} + \frac{kd\,(p-m)}{p(2p-1)}\, T^{(m,\kappa)}_{p-1\,n} \right] \delta_{qm} $$

$$ {}^{e}_{j}\Omega^{1mn(\kappa)}_{2qp}(kd) = \frac{i\,kd\; T^{(m,\kappa)}_{pn}}{p(p+1)} $$

From Curzan find that

$$ {}^{e}_{j}\Omega^{2mn(\kappa)}_{1qp} = {}^{e}_{j}\Omega^{1mn(\kappa)}_{2pq}(kd) \tag{C-17} $$

and

$$ {}^{e}_{j}\Omega^{2mn(\kappa)}_{2qp}(kd) = {}^{e}_{j}\Omega^{1mn(\kappa)}_{1qp}(kd) \tag{C-18} $$

The translation of coordinate system transformation affects only the polar index n and the type index t of $\underset{\sim}{v}^{t(\kappa)}_{mn}$.

APPENDIX D

CONDITIONS FOR TRUNCATING THE MULTIPLE SCATTERING EQUATION

The multiple scattering equation is given by

$$(\underline{\underline{1}} - \underline{\underline{S}})\,\underline{a} = \underline{B} \qquad n = 0, 1, \text{---}\infty$$

where

$$\underline{a} = \{a_n \text{---}\} \qquad n, j = 0, 1, \text{---}\infty$$

$$\underline{\underline{1}} = (\delta_{nj} \text{---}) \qquad n, j = 0, 1, \text{---}\infty$$

$$\underline{\underline{S}} = (S_{nj} \text{---})$$

This multiple scattering equation, to be solved must be truncated at some $n, j = N$. The system of equations can be truncated only when the elements of $\underline{B}$ and $\underline{\underline{S}}$ satisfy certain requirements. These conditions may be determined by considering a formal solution of the equations.

Let

$$\underline{\underline{f}} = \underline{\underline{1}} - \underline{\underline{S}} = (f_{ij}) \qquad i, j \to \infty$$

If the system is truncated at $i = j = N$ then the solution is given by Cramer's rules as

$$|f^{(N)}|\, a_n^{(N)} = \sum_{i=1}^{N} F_{in}^{(N)} B_i$$

where $|f^{(N)}|$ is the determinant of the truncated matrix

$F_{in}^{(N)}$ is the cofactor of the element of the truncated matrix.

This can be expressed as

$$a_n^{(N)} = \frac{\sum_{i=1}^{N} F_{in}^{(N)} B_i}{\sum_{i=1}^{N} F_{in}^{(N)} f_{in}}$$

For $a_n^{(N)}$ to be a solution to the problem it must be related to a_n the exact solution as

$$a_n = a_n^{(N)} = a_n^{(N+1)} = a_n^{(N+J)} \text{ for some N.}$$

The criterion for truncation then can be expressed by considering the solution $a_n^{(N+1)}$ in terms of $a_n^{(N)}$.

$$a_n^{(N+1)} = \frac{\sum_{i=1}^{N+1} F_{in}^{(N+1)} B_i}{\sum_{i=1}^{N+1} F_{in}^{(N+1)} f_{in}}$$

now

$$F_{in}^{(N+1)} = \sum_{j=1}^{N+1} f_{j\,N+1}\, F_{j\,N+1,\,in}^{(N+1)}$$

where $F_{jk,\,in}^{(N+1)}$ is the cofactor of j, k excluding rows i, n computed using the N + 1 extent matrix.

In this form,

$$F^{(N+1)}_{N+1\ N+1,\ in} = F^{(N)}_{in} \quad .$$

Therefore

$$a_n^{(N+1)} = \frac{\sum_{i=1}^{N+1} \sum_{j=1\ j\neq i}^{N+1} f_{j\,N+1}\, F^{(N+1)}_{j\ N+1,\,in}\, B_i}{\sum_{i=1}^{N+1} \sum_{n=1\ j\neq i}^{N+1} f_{j\ N+1}\, F^{(N+1)}_{j\ N+1,\,in}\, f_{in}}$$

$$= \frac{f_{N+1\ N+1} \sum_{i=1}^{N} F^{(N)}_{in} B_i + \sum_{j=1}^{N} \sum_{j=1\ j\neq i}^{N} F_{j\ N+1}\, F^{(N+1)}_{j\ N+1,\,in}\, B_i}{f_{N+1\ N+1} \sum_{i=1}^{N} F^{(N)}_{in} f_{in} + \sum_{j=1}^{N} \sum_{j=1\ j\neq i}^{N} f_{j\ N+1}\, F^{(N+1)}_{j\ N+1,\,in}\, f_{in}} \quad .$$

Now, if we require that $f_{N+1\ N+1} > f_{j\ N+1}$; $f_{N+1\ N+1} > f_{N+1\ j}$;

$$f_{ij} > f_{N+1,\ j};\ f_{in} > f_{j\ N+1} \text{ and } b_j > f_{N+1,\ j} \qquad 0 \leq i,\ j \leq N$$

then the error in truncation is made evident. Let the inequalities be expressed as

$$\frac{F_{N+1\ N+1}}{F_{in}} = \mathcal{O}(1) \qquad k,\ i,\ n \le N$$

$$\frac{F_{k\ N+1}}{F_{in}} = \mathcal{O}(\varepsilon) \quad ; \quad \frac{F_{N+1\ k}}{F_{in}} = \mathcal{O}(\varepsilon)$$

and

$$\frac{n+1}{F_{in}} = \mathcal{O}(\varepsilon) \quad \text{then} \quad \frac{F^{(N+1)}_{j\ N+1,\ in}}{F^{(N)}_{in}} = \mathcal{O}(\varepsilon)$$

The equation for the $a_n^{(N+1)}$ element of the solution is given as

$$a_n^{(N+1)} = \frac{\sum\limits_{i=1} F_{in}^{(N)} B_i + \frac{1}{f_{N+1\ N+1}} \sum\limits_{j=1} \sum\limits_{i=1\ j\neq j} f_{j\ N+1} F^{(N+1)}_{j\ N+1,\ in} B_i}{\sum\limits^{N} F_{in}^{(N)} f_{in} + \frac{1}{f_{N+1\ N+1}} \sum\limits_{j=1}^{N} \sum\limits_{i=1\ i\neq j}^{N+1} f_{j\ N+1} F^{(N+1)}_{j\ N+1,\ in} f_{in}}$$

$$= \frac{\sum\limits_{i=1} F_{in}^{(N)} B_i + \mathcal{O}(\varepsilon^2)}{\sum\limits^{N} F_{in}^{(N)} f_{in} + \mathcal{O}(\varepsilon^2)}$$

$$= a_n^{(N)} + \mathcal{O}(\varepsilon^2)$$

If the inequalities are obeyed by successive value of N, the system can be truncated at some value of N + J with the error less than $\mathcal{O}(\varepsilon^{2J})$ or less than some predetermined value. The matrices $\underline{\underline{S}}$ and $\underline{B}$ obey these requirements if

$$\frac{S_{ij+}}{S_{ij}} < \varepsilon, \quad \frac{S_{i+\ j}}{S_{ij}} < \varepsilon, \quad \text{and} \quad \frac{B_{j+1}}{B_j} < \varepsilon \quad \text{for } j \geq N.$$

If the conditions on S_{ij} and B_j are met, the system of equations as truncated represents the solution to the problem and the problem can be solved by the classical techniques for a large number of simultaneous equations. Two basic solutions are available, one the iterative technique that depends upon $S_{ij} < 1$ for all i, j and the basic matrix inversion technique. The iterative technique is desirable if possible since the errors that may occur in inverting large matrices are minimized. The iterative solution is given by rearranging the matrix equation as

$$\underline{a} = \underline{B} + \underline{\underline{S}}\,\underline{a}$$

and taking the interated solution as

$$\underline{a}^{(i)} = \underline{B} + \underline{\underline{S}}\,\underline{a}^{(i-1)}$$

and using successive i until the results converge. This system converges if the scattering contribution is small compared with the rest. The criterion is given as

$$\sum_{j=1}^{N} |S_{ij}| \ll 1 \quad .$$

If this criterion holds, the system of equations can be solved by iteration. If not, the system must be treated by matrix inversion.

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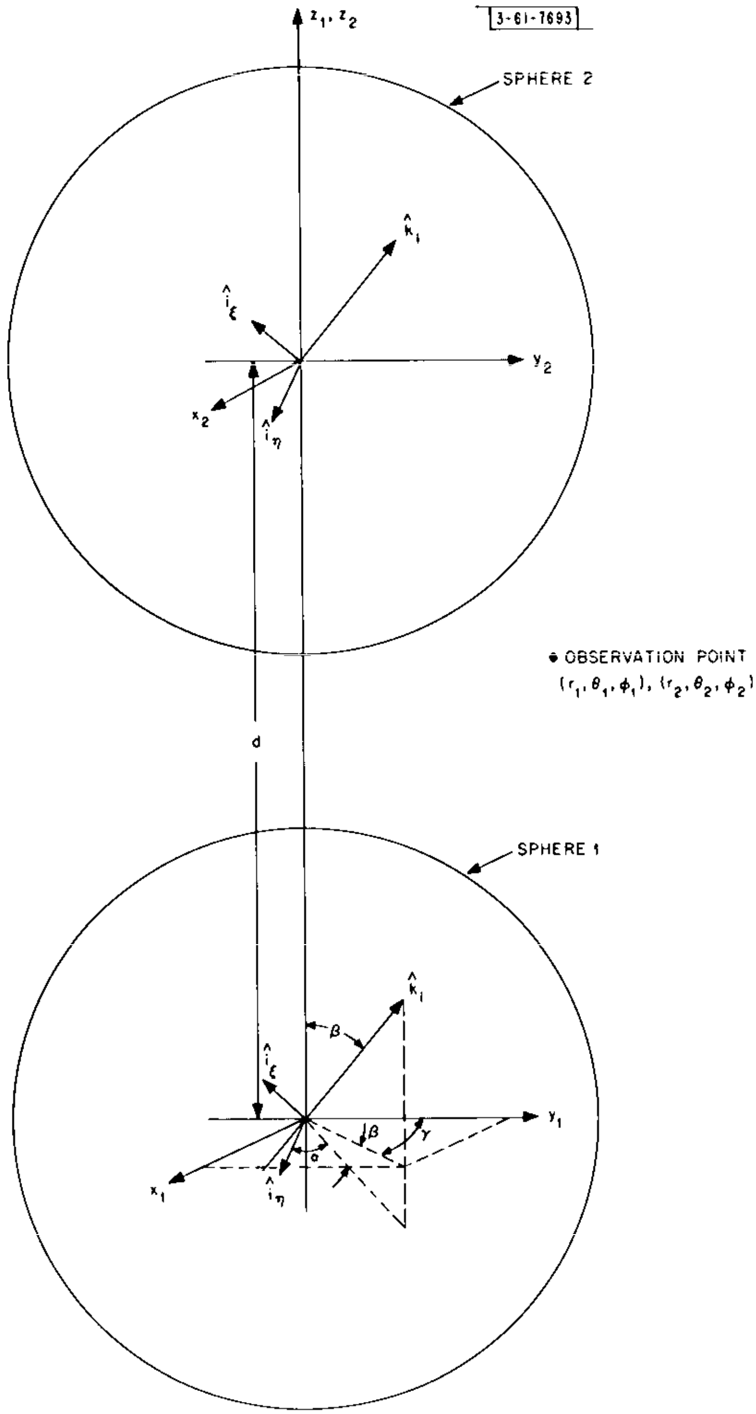

Fig. 1. Geometry of the two-sphere problem.

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Security Classification

DOCUMENT CONTROL DATA – R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Lincoln Laboratory, M.I.T.	2a. REPORT SECURITY CLASSIFICATION Unclassified	
	2b. GROUP None	
3. REPORT TITLE Cooperative Scattering by Dielectric Spheres		
4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)* Technical Note		
5. AUTHOR(S) *(Last name, first name, initial)* Crane, Robert K.		
6. REPORT DATE 6 July 1967	7a. TOTAL NO. OF PAGES 66	7b. NO. OF REFS 13
8a. CONTRACT OR GRANT NO. AF 19(628)-5167	9a. ORIGINATOR'S REPORT NUMBER(S) Technical Note 1967-31	
b. PROJECT NO. 649L		
c.	9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)* ESD-TR-67-326	
d.		
10. AVAILABILITY/LIMITATION NOTICES This document has been approved for public release and sale; its distribution is unlimited.		
11. SUPPLEMENTARY NOTES None	12. SPONSORING MILITARY ACTIVITY Air Force Systems Command, USAF	

13. ABSTRACT

The problem of scattering of electromagnetic waves by a small number of closely spaced dielectric spheres is considered as a boundary value problem. The solution to this problem is obtained in a series form using partial spherical vector waves. An approximate solution is also obtained for spheres separated sufficiently far for waves scattered by one sphere and incident on another to be considered plane waves with an amplitude given by the solution to the single scattering problem. The use of both solutions is discussed.

14. KEY WORDS

scattering
electromagnetic waves
dielectric spheres
plane waves

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Security Classification